Algorithms over Dynamic Graphs

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Abstract. Graph theory provides mathematical models with computational realizations for a wide range of problems. The classic version provides static models and solutions for these problems. These solutions are often insufficient to versions of the problems in wich the information changes with respect to a continuous variable, eg. time. In the same way one can think on dynamic graphs as graphs in wich some components (arcs, edges, costs) change with respect to a continuous variable. This paper explores graphs with dynamic costs as a bundle, and includes the formulation and the solution for the shortest path problem and the maximum flow problem on these structures. This paper provides an unexplored connection between the dynamic graph theory and the topology, presents approaches to the solution of dynamic versions for the shortest path problem and the maximum flow problem, and proposes both a new source of applications of the metric bundles theory and the type two theory of effectivity.

Keywords: Bundles of metric spaces, graphs with dynamic costs, dynamic graphs.

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1 Introduction

Graph theory provides mathematical models with computational realizations for a wide range of problems. The classic version provides static models and solutions for central problems in graph theory: shortest path problem (SPP), minimum spanning tree (MST), maximum flow problem (MFP) and many others. The static solutions are, for example, the Dijkstra algorithm to find the shortest path between two nodes in a connected graph with nonnegative costs [4], the Ford-Fulkerson algorithm to find the maximum flow between two nodes in a graph with limited capabilities in edges [6], the Kruskal's algorithm [10] to find the minimal spanning tree in a connected graph, etc.

Obviously, one of the complications offered by the reality to these problems is related with to the fact that the real information is constantly changing. For example, the length of the roads in a city is not enough information to decide the route that will consume less time to go from the home to the workplace in the morning, and the most efficient path in a particular time is often not in another time.

In general, the graphs in that part of the information changes with respect to a variable are called dynamic graphs. This paper is limited to dynamic graphs where the costs in the edges change with respect to a continuous variable, for example, time: graphs with dynamic costs or dynamically costed graphs.

The paper is structured as follows: section 2 is devoted to introduce the basic concepts that will be used in the paper, in section 3 we introduces the concept of graphs with dynamic costs and built the metric bundle associated with a graph of this nature, in section 4 we presents the shortest path problem in a graph with dynamic costs and solutions are proposed, section 5 presents the related work on the maximum flow problem on dynamic graphs. To conclude, section 6 presents conclusions and future work. Additionally, appendix 7 presents the basic concepts of bundles of metric spaces or metric bundles.

2 Basic Concepts

The first part of this section is devoted to presents the basic concepts of graph theory. [3] is one of the many references for further in graph theory. The second part presents the basic concepts of topology. The recommended reference is [17].

2.1 Graph Theory

Definition 1 (Graph) *A graph is a pair $G = \langle V_G, E_G \rangle$ such that V_G is a set and $E_G \subseteq V_G \times V_G$. If $v \in V_G$ then v is a vertex (node) in G and, if $e \in E_G$ then e is an edge (arc) in G. If $c_G : E_G \longrightarrow \mathbb{R}$ is a function, then $G = \langle V_G, E_G, c_G \rangle$ is a costed graph and c is the costs function of G.*

Throughout this paper we will assume that G is a finite graph unless otherwise stated, i.e. V is a finite set.

Definition 2 (Graph isomorphism) *ψ is graph isomorphism between $G = \langle V_G, E_G \rangle$ and $H = \langle V_H, E_H \rangle$, if ψ is a function such that*

1. *$\psi : V_G \longrightarrow V_H$ is a bijection, and*
2. *for all $(v_1, v_2) \in V_G \times V_G$, $e = (v_1, v_2) \in E_G$ if and only if $e' = (\psi(v_1), \psi(v_2)) \in E_H$.*

In this case, we say that G and H are isomorphic graphs and write $G \sim H$.

Note that an isomorphism ψ induces a bijection $\psi_E : E_G \longrightarrow E_H$ defined by $\psi_E((u, v)) = (\psi(u,)\psi(v))$.

Definition 3 (Path between two nodes) *Let G be a graph. If v_1 and v_2 are nodes in V_G and $\beta = \langle w_1 \dots w_n \rangle$ is a list of nodes in V_G, we say that β is a path between v_1 and v_2 if and only if*

1. *$w_1 = v_1$,*
2. *$w_n = v_2$, and*
3. *$(w_i, w_{i+1}) \in E_G$ for all $i = 1, \dots, n-1$.*

If all nodes in β are different, then we say that β is a simple path between v_1 and v_2.

If G is a costed graph, then c_G can be extended to all paths in G: If $\beta = \langle w_1 \dots w_n \rangle$ is a path between w_1 and w_2, then

$$c_G(\beta) = \sum_{i=1}^{n-1} c_G((w_i, w_{i+1})) \quad (1)$$

Definition 4 (Adjacency set) *Let $G = \langle V_G, E_G \rangle$ be a graph, and $v \in V_G$ be a vertex in G. The adjacency set of v is the set of all vertex in G that are connected directly with v, that is*

$$adj(v) = \{w | (v, w) \in E_G\}. \quad (2)$$

2.2 Topology

Definition 5 (Topological space) *The tuple $\mathfrak{X} = (X, \tau)$ is a topological space if and only if X is a set and τ is a family of subsets of X such that*

- *$X \in \tau$,*
- *$\emptyset \in \tau$,*
- *X is closed under arbitrary unions, in other words, if $\beta \subseteq \tau$ then $\bigcup_{B \in \beta} B \in \tau$, and*
- *X is closed under finite intersections, in other words, if $B_1, \dots, B_n \in \tau$ then $\bigcap_{k=1}^{n} B_k \in \tau$.*

In this case we say that τ is a topology over X, and if $U \in \tau$ we say that U is an open set in X.

Definition 6 (Base for a topology) *A base for τ is a class $\mathcal{B}$ of open sets with the property that every open set is a union of sets in $\mathcal{B}$.*

Definition 7 (Continuous functions) *Let (X, τ_X) and (Y, τ_Y) be two topological spaces, and $f : X \longrightarrow Y$ be a function, we say that f is a continuous function if and only if, $f^{-1}(V) \in \tau_X$ whenever $V \in \tau_Y$.*

Definition 8 (Metric space) *$\mathfrak{X} = (X, d)$ is a metric space if X is a set and $d : X \times X \longrightarrow \mathbb{R}^+$ is a function such that, for all $x, y, z \in X$,*

- *$d(x, y) = 0 \Longleftrightarrow x = y$,*
- *$d(x, y) = d(y, x)$, and*
- *$d(x, y) + d(y, z) \leq d(x, z)$.*

Every metric space can be viewed as a topological space with the topology generated by the base

$$\{B_r(x) | x \in X \wedge r \in \mathbb{R}^+\}$$

where

$$B_r(x) = \{y | d(x, y) < r\}.$$

3 Dynamically costed graphs

In general, a dynamic graph is a graph in which part of the information changes with respect to a variable. This paper is limited to dynamic graphs in wich the arc costs change with respect to a variable. Formally, we assume that $G = \langle V_G, E_G \rangle$ is a graph, T is a topological space, and, for all $t \in T$, $c_t : E_G \longrightarrow \mathbb{R}^+$ is a cost function over the arcs of G. In other words, for all $t \in T$, $G = \langle V_G, E_G, c_t \rangle$ is a costed graph.

Definition 9 (Dynamically costed graphs) *Let G be a graph, and let T be a topological space. Let C be a familly, inexed by T, of cost functions over the arcs of G. In other words, for all $t \in T$, there exists a unique function $c_t : E_G \longrightarrow \mathbb{R}^+$ in C such that $G = \langle V_G, E_G, c_t \rangle$ is a costed graph. If, for all edge $e \in E_G$, the function $k_e : T \longrightarrow \mathbb{R}^+$ defined by $k_e(t) = c_t(e)$ is continuous, then we say that $\mathfrak{G} = (G, C, T)$ is a graph with dynamic costs or dynamically costed graph.*

In general, unless otherwise stated, we assume that all elements of C take nonnegative values.

If $\mathfrak{G} = (G, C, T)$ is a dynamically costed graph, and $\beta = \langle w_k \rangle_{k=0}^{q}$ is a path, then we say that the cost of β in $\mathfrak{G}$ is defined by

$$c_{\mathfrak{G}}(\beta) = \sum_{k=0}^{q-1} c_{t_k}\left((w_k, w_{k+1})\right) \tag{3}$$

where $t_0 = 0$, and $t_k = t_{k-1} + c_{t_{k-1}}\left((w_{k-1}, w_k)\right)$ for $k = 1, \ldots, q-1$.

3.1 Graphs with dynamic costs as bundles of metric spaces

Check for the appendix 7 for concepts on bundles of metric spaces.

Henceforth it is understood that $\mathfrak{G} = (G, C, T)$ is a graph with dynamic costs such that $G = \langle V_G, E_G \rangle$ is connected.

Note that in these conditions, for all $t \in T$, it is possible to define the function $d_t : V \times V \longrightarrow \mathbb{R}^+$ such that $d_t(v_1, v_2)$ is the length of the shortest path between the nodes v_1 and v_2 in G with respect to the cost function c_t. d_t can be calculated, for example, by Dijkstra's algorithm [4] or the Floyd-Warshall algorithm.

Proposition 1 *For all $t \in T$, $\mathfrak{V} = (V_G, d_t)$ is a metric space.*

The following theorem is an application of the Existence Theorem for bundles of metric spaces presented in [7].

Theorem 1 *Let $E = V_G \times T$, $\pi_2 : E \longrightarrow T$ the second projection $\pi_2(v, t) = t$, $\Sigma = \{\alpha_v : v \in V_G\}$ with $\alpha_v : T \longrightarrow E$ defined by $\alpha_v(t) = (v, t)$ for all $t \in T$, and the function $d : E \times E \longrightarrow [0, +\infty]$ defined by $d\left((v_1, t_1), (v_2, t_2)\right) = +\infty$, if $t_1 \neq t_2$ and $d\left((v_1, t_1), (v_2, t_2)\right) = d_t(v_1, v_2)$ if $t_1 = t_2 = t$. Then (E, π_2, T) is a bundle of metric spaces, Σ is a full set of global sections for π_2 and the family of all ϵ-tubes around $\alpha_v \restriction_U$, where $\epsilon > 0$, α_v runs Σ and U through the collection of nonempty open sets in T, is a base for the topology of E.*

Proof. It is clear that d is a metric for π_2 and, since for every $v \in V$, $\pi_2(\alpha_v(t)) = \pi_2(v, t) = t$, the collection Σ constitutes a family of selections for π_2. Furthermore, if $(v, t) \in E$, then $(v, t) = \alpha_v(t)$, and thus $(v, t) \in \mathcal{T}_\epsilon(\alpha_v)$ for every $\epsilon > 0$.

Let $v_1, v_2 \in V$ and define $\Phi_{v_1 v_2} : T \longrightarrow [0, +\infty]$ by

$$\Phi_{v_1 v_2}(t) = d\left(\alpha_{v_1}(t), \alpha_{v_2}(t)\right).$$

Since $\pi_2\left(\alpha_{v_1}(t)\right) = \pi_2\left(\alpha_{v_2}(t)\right)$, we must have

$$\Phi_{v_1 v_2}(t) = d\left(\alpha_{v_1}(t), \alpha_{v_2}(t)\right) \neq +\infty$$

and

$$\Phi_{v_1 v_2}(t) = d\left(\alpha_{v_1}(t), \alpha_{v_2}(t)\right) = d_t(v_1, v_2).$$

For every simple path, $\beta = \langle w_k \rangle_{k=0}^{q}$, from v_1 to v_2 and for every $t \in T$, let

$$c_t(\beta) = \sum_{i=0}^{q-1} c_t\left((w_i, w_{i+1})\right)$$

be the cost of path β over the graph $G_t = \langle V_G, E_G, c_t \rangle$, that is, the cost of path β with regard to the cost function c_t.

Under these conditions, given that $k_{(w_i, w_{i+1})}$ is a continuous function, that is to say,

$$\lim_{u \to t} c_u\left((w_i, w_{i+1})\right) = c_t\left((w_i, w_{i+1})\right),$$

we must have

$$\lim_{u \to t} c_u(\beta) = \lim_{u \to t} \sum_{i=0}^{q-1} c_t\left((w_i, w_{i+1})\right) = c_t(\beta).$$

So that if $\beta_1, \ldots, \beta_m$ are paths starting at v_1 and ending at v_2, the function

$$F_{\beta_1, \ldots, \beta_m}(t) = \min_{i=1,\ldots,m} \{c_t(\beta)\}$$

is continuous on t.

To complete the hypthesis of the theorem for Existence of Bundles of Metric Spaces [7], it is enough to notice that if $P(v_1, v_2)$ is the set of paths connecting v_1 and v_2, $P(v_1, v_2)$ is finite. Thus,

$$\Phi_{v_1 v_2}(t) = \min_{\beta \in P(v_1, v_2)} \{c_t(\beta)\}$$

is continuous. This allows us to conclude that $\mathbb{G} = (V_G \times T, \pi_2, T)$ is a bundle of metric spaces , Σ is a full set of global sections over for π_2 and that the collection of ϵ-tubes around $\alpha_v \upharpoonright_U$, for $\epsilon > 0$, $\alpha_v \in \Sigma$ and U a non-empty open set of T, constitute a basis for the topology of E. $\square$

Definition 10 *The bundle of metric spaces* $\mathbb{G}$ *constructed previously is known as the* bundle of metric spaces associated with the dynamically costed graph $\mathfrak{G}$*, or the* Bundle of Metric Spaces of $\mathfrak{G}$.

Proposition 2 *Let* $t_1, t_2 \in T$ *and denote by* $\pi_2^{-1}(t_1)$ *and* $\pi_2^{-1}(t_2)$ *their corresponding fibers in* $\mathbb{G}$*. In this case*

$$G_{t_1} = \left\langle \pi_2^{-1}(t_1), E_G, c_{t_1} \right\rangle$$

and

$$G_{t_2} = \left\langle \pi_2^{-1}(t_2), E_G, c_{t_2} \right\rangle$$

and in this case, both G_{t_1} *and* G_{t_2} *are isomorphic graphs.*

4 Shortest path in dynamically costed graphs

A problem that cannot be obviously modelled using statically-costed graphs is the problem of traffic distribution. Indeed, finding the shortest path between two locations in a city along a -directed- network of roads, must take into account phenomena such as rush-hours, traffic jams, and works on the roads.

This problem can be modelled in a particular situation by introducing a *costed* graph G, onto which to find the shortest path. Conceptually, this is done letting $G = \langle V, E \rangle$ to have a function $c : E \longrightarrow \mathbb{R}^+$ assigning the *cost* of traversing edge (i.e. street) e. The problem of finding the shortest route between vertices v_1 and v_2 can be described as the problem of finding a path $\beta = \langle w_k \rangle_{k=0}^q$, from v_1 to v_2, in such a way that $c(\beta)$ is minimal among all paths connecting node v_1 and node v_2.

The static version can be formulated as follows:

- **Instance:** A conected graph $G = \langle V_G, E_G \rangle$, a costs fuction $c_G : E_G \longrightarrow \mathbb{R}^+$, and two nodes $v_1, v_2 \in V_G$.
- **Answer:** A path $\beta = \langle w_k \rangle_{k=0}^q$, from v_1 to v_2 in G, such that $c_G(\beta) \leq c_G(\gamma)$, for all path $\gamma = \langle u_k \rangle_{k=0}^{p_\gamma}$, from v_1 to v_2.

An algorithmic solution to the shortest-path problem in static graphs is known and can be calculated using a variety of techniques for example Dijkstra's algorithm [4]. However, in order to take into account the problem of time-varying costs of traversing a path (in the sense of time spent), dynamically costed graphs must be introduced. In this instance, the *cost function* c of graph G varies along time: $c : E \times \mathbb{R}^+ \to \mathbb{R}$.

This situation can be explored in a more general setting, allowing the description and exploration of shortest-path-problems in -for example- traffic.

Indeed, shortest-path in dynamically costed graphs is a problem that can be tackled using bundles of metric spaces (see appendix 7), by regarding the graph itself as a bundle of metric spaces:

$$\mathbb{G} = \left(V_G \times \mathbb{R}^+, \pi_2, \mathbb{R}^+\right),$$

where graph G can be written as $G = \langle V_G, E_G \rangle$ and is required to be connected. The problem of finding the shortest path between nodes v_1 and v_2 in a dynamically costed graph can be formulated as follows:

- **Instance:** A graph with dynamic costs or dynamically costed graph $\mathfrak{G} = (G, C, \mathbb{R}^+)$ where $G = \langle V_G, E_G \rangle$ is a connected graph, and two nodes v_1, $v_2 \in V_G$.
- **Answer:** A path $\beta = \langle w_k \rangle_{k=0}^q$, from v_1 to v_2 in G, such that $c_{\mathfrak{G}}(\beta) \leq c_{\mathfrak{G}}(\gamma)$, for all path $\gamma = \langle u_k \rangle_{k=0}^{p_\gamma}$, from v_1 to v_2.

4.1 Traffic distribution problem

Consider the situation of finding the shortest path between two points A and B in a city, whose street structure is describable by a costed graph $G = \langle V, E, l \rangle$ where the costs function l is associate to the lenght of streets: $l(e)$ is the length of the street e. In order to drive from point A to point B we define for each street e, the average velocity on e on time instant t by $v_e(t)$.

In this sense, the time spent in traversing path e at time t, $c_t(e)$, is given by

$$\int_t^{c_t(e)} v_e(s)ds = l(e). \tag{4}$$

The time cost of traversing a path $\beta = \langle w_k \rangle_{k=0}^{p}$ is given by

$$c(\beta) = \sum_{k=0}^{p-1} c_{t_k}((w_k, w_{k+1})) \tag{5}$$

where $t_0 = 0$ and $t_k = t_{k-1} + c_{t_{k-1}}((w_{k-1}, w_k))$ for $k = 1 \ldots p-1$.

Notice that the problem posed in this way implies a *non-overpass* condition: vehicles cannot cut ahead of one another and must finish the path in the same ordering as they enter. Formally speaking, for every edge $e \in E_G$ and every pair of time instants $t_0, t_1 \in \mathbb{R}^+$ where $t_0 < t_1$ we must have $c_e(t_0) \leq c_{t_1}(e) + t_0$.

In terms of bundles of metric spaces $\mathbb{G}$ of $\mathfrak{G}$, the existence of the number

$$\tau = c_t((u, v)) \neq +\infty,$$

represents the possibility of reach the point $(v, t + \tau)$ from the point (u, t).

In this sense, $\mathbb{G}$ can be associated to an infinite graph $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ where:

- $V_{\mathcal{G}} = V \times \mathbb{R}^+$ and
- $((u, t), (v, t')) \in E_{\mathcal{G}}$ if and only if $(v, v) \in E$ and $t' = t + c_t((u, v))$.

A characterization of the arcs present in graph $\mathcal{G}$ that is useful comes in terms of the adjacency set of a vertex $(u, t) \in V_{\mathcal{G}} = V \times \mathbb{R}^+$. Indeed, by defining the set

$$\begin{aligned} & adj((u,t)) \\ = & \\ & \{(w, t') \mid (u, w) \in E \wedge t' = t + c_t((u, w))\}. \end{aligned} \tag{6}$$

To calculate the time-cost of a minimal-cost path between nodes u and w in $\mathfrak{G}$ at time instant t is equivalent to find the minimal value of $\tau > 0$ for which

$$((u, t), (w, t + \tau)) \in \mathcal{G}$$

This problem can be solved, when posed like this, by either introducing a classical *sideways-search* algorithm in $\mathcal{G}$ starting from point (u, t) or by using dynamic programming and recursively calculating the expression

$$\begin{aligned} & D_t(u, w) \\ = & \\ & \min_{r \in adj(u)} \left\{ c_t(u, r) + D_{t + c_t(u,r)}(r, w) \right\}, \end{aligned} \tag{7}$$

with $D_t(w, w) = 0$, to find the minimal cost path between nodes u and w in $\mathfrak{G}$.

5 Maximum flow in dynamically costed graphs

When simulating transport phenomena using graphs, the concept of flow arises in a very natural sense. The amount of material (particles, vehicles or load) that the graph can transport. The problem of maximum flow in classical graph theory, deals with finding the the largest amount of material that can be sent between a node known as the *source* and a node known as the *destination*.

More formally, if G is a graph (or network) with n vertices, where the first vertex is regarded as the source, and the last one as the destination. If the capacity associated to every edge (i, j) is denoted by w_{ij}, we need to determine the collection of values x_{ij} (for $1 \leq i < n$ and $1 < j \leq n$), such that

- for all $1 \leq i < n$ and $1 < j \leq n$, $0 \leq x_{ij} \leq w_{ij}$,
- for all vertex $1 < i < n$,

  $$\sum_{k=1}^{n-1} x_{ki} = \sum_{k=1}^{n-1} x_{ik}, \tag{8}$$

  and
- the sum

  $$\sum_{k=2}^{n} x_{ik} = \sum_{k=1}^{n-1} w_{kn} = F \tag{9}$$

  is maximal with regard to every other set of values.

In addition to the solution offered by linear programming, this problem has been explored in classical graph theory, providing solutions as the Ford-Fulkerson and the Edmons-Karp algorithms [6, 5].

A dynamic version of the maximum flow problem tries to determine the maximum amount of material that can be sent throughout a dynamic network G in a given time span $[t_0, t_1]$ from the source to the destination, assuming that the capacity of the edges changes continually according to functions $w_{ij}(t)$.

The solution to the maximum flow problem in a dynamic graph can be estimated by defining a function $F : [t_0, t_1] \longrightarrow \mathbb{R}^+$ such that, for every $t \in [t_0, t_1]$, $F(t)$ is the value of the maximum flow of graph G at time instant t using for example the Ford-Fulkerson algorithm. The maximum flow along time period $[t_0, t_1]$ can be found using the integral

$$\int_{t_0}^{t_1} F(t)dt \tag{10}$$

In order to produce an algorithm approximating the solution to this problem, we define a partition $\{u_i\}_{i=0}^n$ of interval $[t_0, t_1]$, $t_0 = u_0 < u_1 < \ldots < u_{n-1} < u_n = t_1$, and we execute an algorithm to solve the problem of maximal flow on each time instant u_i. If we denote by F_i the value of function F at time instants u_i, we can approximate the integral on equation 10 using -among others- Simpson's rule

$$\begin{array}{c} \int_{t_0}^{t_1} F(t)dt \\ = \\ \frac{1}{2}\sum_{i=0}^{n-1}\left(F\left(u_i\right)+F\left(u_{i+1}\right)\right)\left(u_{i+1}-u_i\right). \end{array} \tag{11}$$

6 Conclusions and Future Work

Under the presentation made, the problem of finding the shortest path between nodes on a dynamic graph can be proven to be solvable in polynomial time -whenever conditions of non-overpass are met- [11]. Moreover, the structure and concepts presented in this paper allow the exploration of the generalized shortest-path problem under a new perspective, it is important to notice that this problem is actually NP-hard [12]. In turn, maximum flow in dynamic graphs can be regared to have a complexity of

$$\mathcal{O}\left(|E_G| \max_{i=0,\ldots,n}\left(\max_{e\in E_G} w_{u_i}(e)\right)\right)$$

with the solution presented here. Furthermore, dynamic graphs can be analyzed using bundles of metric spaces. This induces new perspectives of both the theory of graphs with dynamic costs and introduces new applications of bundles of metric spaces. To be more concrete, this new approximation on the theory of graphs with dynamic costs in such a way that:

1. it establishes an unexplored connection between the theory of graphs with dynamic costs and computable topology, under which several new applications of the theory of Bundles of metric spaces may flourish, and

2. it proposes a new source of ideas and induces unexplored applications for Type-2 Theory of effectivity (TTE for short), when conditions of computability are imposed on either the base or the fiber space (see [8, 16]).

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7 Appendix. Bundles of metric spaces.

According to [2] and [7], the category of bundles of metric spaces over a given topological space T can be thought of as a generalization of the category of metric spaces over a field -conveniently named T-. In this appendix we present some basic results about Bundles of Metric Spaces. For concepts, proofs and more results on this topic we encourage the reader to revise sources as J.Varela [14, 15].

Definition 11 (Sections and selections) *Let G be a set, X be a topological space and $p : G \longrightarrow X$ be a surjective function. A* selection *for p is a function $\alpha : A \subseteq X \longrightarrow G$ such that $p \circ \alpha$ is the identity function over $Q = dom(\alpha)$. If $Q = X$ we say that α is a* global selection*. If α is continuous, we say that α is a* section*.*

Given a set Γ of selections over a function $p : G \longrightarrow X$ we say that collection Γ is *full* if for every $u \in G$ we can find $\alpha \in \Gamma$ such that $\alpha(p(u)) = u$.

Definition 12 (Metric for a surjective function) *Let G be a set, X be a topological space and $p : G \longrightarrow X$ be a surjective function. A function $d : G \times G \longrightarrow [0, +\infty]$ is called a* metric for function *p if for every $u, v, w \in G$:*

1. $d(u, v) = +\infty$ *if* $p(u) \neq p(v)$
2. $d(u, v) = 0$ *if and only if* $u = v$
3. $d(u, v) \leq d(u, w) + d(w, v)$

Provided with a metric for a function, we can define ϵ-tubes around selections, that can be used as basic open subsets to generate topologies. An ϵ-tube around a selection α of radius is defined as the set

$$\mathcal{T}_r(\alpha) = \{u \in G : \ p(u) \in \mathrm{dom}(\alpha) \wedge d(u, \alpha(p(u))) < \epsilon\},$$

the idea behind constructing bundles of metric spaces is to generalize the notion of metrizability. This generalization is condensed in the following definition:

Definition 13 (Bundle of metric spaces) *Let G and X be topological spaces, $p : G \longrightarrow X$ a continuous surjective function. If d is a metric for p such that for every $u \in G$ there exists a local selection α for p, such that $u \in \mathcal{T}_\epsilon(\alpha)$ for some $\epsilon > 0$, we say that the triplet (G, p, T) is a bundle of metric spaces, whenever the collection of all ϵ-tubes defines a topology over G.*

In the language of bundles of metric spaces, the set X is called the *base space* and has the most structure in terms of topological properties. Space G is called the *fiber space*, since we can write

$$G = \bigcup_{x \in X} p^{-1}(x)$$

where each $G_x = p^{-1}(x)$ is called a *fiber* above $x \in X$.

7.1 Existence of Bundles of Metric Spaces: a Theorem

We wish to present a modified version of a central result in computable analysis: an existence theorem for bundles of metric spaces as proven in [1]. This theorem stablishes conditions under which metric spaces can occur and its computable version allows the introduction of computability over a bundle, provided with minimal conditions on the base space.

Theorem 2 (Existence of bundles of metric spaces) *If X is a topological space, β_X a basis for the topology of X, G a non-empty set of at most continuum cardinality and $p : G \longrightarrow X$ a surjective function. Let also d be a metric for p and Γ a collection of local selections for p. Assume also that*

1. *For every $u \in G$ and every $\epsilon > 0$, there exists a local selection $\gamma \in \Gamma$ and a rational number $r \in \mathbb{Q} \cap (0, \epsilon)$ such that $u \in \mathcal{T}_r(\gamma)$.*
2. *For every $\gamma, \zeta \in \Gamma$, the function*

   $$\Phi_{\gamma\zeta} : dom(\gamma) \cap dom(\zeta) \longrightarrow \mathbb{R}$$

   defined by

   $$\Phi_{\gamma\zeta}(p) = d(\gamma(p), \zeta(p))$$

   is upper-semi-continuous

Then G can be given a topology $\mathfrak{T}$ in such a way that

1. *The collection $\beta_G = \{\mathcal{T}_r(\gamma_Q)\}$, where $r \in \mathbb{Q} \cap (0, +\infty)$, $\gamma \in \Gamma$, $Q \subseteq dom(\gamma)$ and $Q \in \beta_X$, is a basis for the topology $\mathfrak{T}$ over G.*
2. *Under the topology $\mathfrak{T}$, Γ is a family of local* sections *over p*
3. *(G, p, T) is a bundle of metric spaces.*

For a proof of the this theorem, and for basic references on this topic, the reader is encouraged to revise [2] and [7].