A Linear-Time Algorithm for k-Partitioning Doughnut Graphs

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Abstract. Given a graph $G = (V, E)$, k natural numbers $n_1, n_2, ..., n_k$ such that $\sum_{i=1}^{k} n_i = |V|$, we wish to find a partition $V_1, V_2, ..., V_k$ of the vertex set V such that $|V_i| = n_i$ and V_i induces a connected subgraph of G for each i, $1 \leq i \leq k$. Such a partition is called a k-partition of G. The problem of finding a k-partition of a graph G is NP-hard in general. It is known that every k-connected graph has a k-partition. But there is no polynomial time algorithm for finding a k-partition of a k-connected graph. In this paper we give a simple linear-time algorithm for finding a k-partition of a "doughnut graph" G.

Keywords: Planar Graph, Doughnut Graph, Graph Partitioning, Hamiltonian-connected.

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1 Introduction

Given a graph $G = (V, E)$, k natural numbers n_1, n_2, ..., n_k such that $\sum_{i=1}^{k} n_i = |V|$, we wish to find a k-partition $V_1, V_2, \ldots, V_k$ of the vertex set V such that $|V_i| = n_i$ and V_i induces a connected subgraph of G for each $i, 1 \leq i \leq k$. A k-partition of a graph G is illustrated in Figure 1 for k = 5 where the edges of five connected subgraphs are drawn by solid lines, and the remaining edges of G are drawn by dotted lines. Let

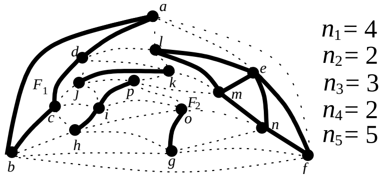

Figure 1: A 5-partition of a 5-connected planar graph G.

$B = u_1, u_2, ..., u_m$ be a sequence of distinct vertices of G with $m \leq k$. A *k-partition of G with basis B* is a k-partition with the additional restriction that $u_i \in V_i$, for $1 \leq i \leq m$. A k-partition of a graph G with basis m is illustrated in Figure 2 for $k = 5$ and $m = 5$.

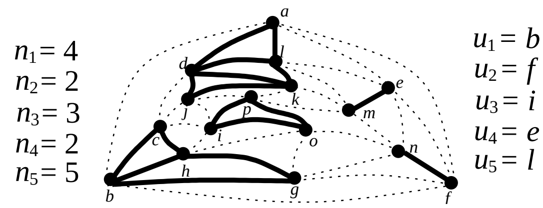

Figure 2: A 5-partition of a 5-connected planar graph G with basis 5.

The problem of finding a k-partition of a given graph often appears in the load distribution among different power plants and the fault-tolerant routing of communication networks [10, 9]. The problem is NP-hard in general even k is limited to 2 [2], and hence it is very unlikely that there is a polynomial-time algorithm to solve the problem. Although not every graph has a k-partition, Györi and Lovász independently proved that every k-connected graph has a k-partition [4, 8]. However, their proofs do not yield any polynomial-time algorithm for finding a k-partition of a k-connected graph. A linear-time algorithm is known for 4-partitioning of a 4-connected plane graph if the four basis vertices are all on the boundary of one face [10]. A linear-time algorithm is also known for 5-partitioning of a 5-connected internally triangulated plane graph if the five basis vertices are all on the boundary of one face [9].

In this paper we give a linear-time algorithm for finding a k-partition of a "doughnut graph" G. The class of "doughnut graphs" is an interesting class of graphs which was recently introduced in graph drawing literature for it's beautiful area-efficient drawing properties [6, 11, 12, 7]. Our algorithm is also applicable for finding a k-partition of a "doughnut graph" with basis at most two. Using the same method, one can find a k-partition of a 4-connected planar graph in linear time.

The rest of the paper is organized as follows. Section 2 describes some of the definitions used in this paper. In Section 3, we give an algorithm for finding a Hamiltonian path between any pair of vertices of a doughnut graph. Section 4 provides a linear-time algorithm for finding a k-partition of a doughnut graph. Finally Section 5 concludes the paper.

2 Preliminaries

In this section we give some definitions.

Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. Throughout the paper, we denote by n the number of vertices in G, that is, $n = |V|$, and denote by m the number of edges in G, that is, $m = |E|$. An edge joining vertices u and v is denoted by (u, v). The degree of a vertex v, denoted by $d(v)$, is the number of edges incident to v in G. G is called *r-regular* if every vertex of G has degree r. We call a vertex v a *neighbor* of a vertex u in G if G has an edge (u, v). The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. G is called *k-connected* if $\kappa(G) \geq k$. We call a vertex of G a *cut-vertex* of G if its removal results in a disconnected or single-vertex graph. For $W \subseteq V$, we denote by $G - W$ the graph obtained from G by deleting all vertices in W and all edges incident to them. A *cut-set* of G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component or $G - S$ is a single vertex graph. A *path* in G is an ordered list of distinct vertices $v_1, v_2, ..., v_q \in V$ such that $(v_{i-1}, v_i) \in E$ for all $2 \leq i \leq q$. Let $P_1 = x_i, \ldots, x_k$ and $P_2 = x_m, \ldots, x_o$ be two paths. We denote by P_1P_2 the concatenation of two paths P_1 and P_2 where the last vertex of P_1 and the first vertex of P_2 are adjacent, i.e, $P_1P_2 = x_i, \ldots, x_k, x_m, \ldots, x_o$ where x_m is a neighbor of x_k.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed embedding. A plane graph G divides the plane into connected regions called $faces$. The unbounded region is called the *outer face*. Let $v_1, v_2, ..., v_l$ be all the vertices in a clockwise order on the contour of a face f in G. We often denote f by $f(v_1, v_2, ..., v_l)$. For a face f in G we denote by $V(f)$ the set of vertices of G on the boundary of face f. We call two faces F_1 and F_2 are *vertex-disjoint* if $V(F_1) \bigcap V(F_2) = \emptyset$.

Let G be a 5-connected planar graph, let Γ be any planar embedding of G and let p be an integer such that $p \geq 3$. We call G a *p-doughnut* graph if the following Conditions (d_1) and (d_2) hold:

(d_1) Γ has two vertex-disjoint faces each of which has exactly p vertices, and all the other faces of Γ has exactly three vertices; and

(d_2) G has the minimum number of vertices satisfying Condition (d_1).

In general, we call a p-doughnut graph for $p \geq 3$ a *doughnut graph*. Figure 3(i) illustrates a doughnut graph. The following result is known for doughnut graphs [6].

Lemma 2.1 *Let G be a p-doughnut graph. Then G is 5-regular and has exactly $4p$ vertices.*

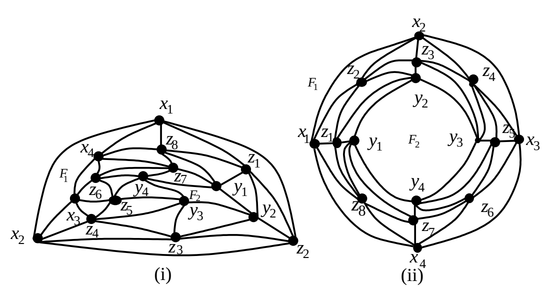

Figure 3: (i) A p-doughnut graph G where $p = 4$ and (ii) a doughnut embedding of G.

For a cycle C in a plane graph G, we denote by $G(C)$ the plane subgraph of G inside C excluding C. Let C_1, C_2 and C_3 be three vertex-disjoint cycles in a planar graph G such that $V(C_1) \cup V(C_2) \cup V(C_3) = V(G)$. Then we call a planar embedding Γ of G a *doughnut embedding* of G if C_1 is the outer face and C_3 is an inner face of Γ, $G(C_1)$ contains C_2 and $G(C_2)$ contains C_3. We call C_1 the *outer cycle*, C_2 the *middle cycle* and C_3 the *inner cycle* of Γ. Figure 3(ii) illustrates a doughnut embedding of the doughnut graph in Figure 3(i). The following result is also known for doughnut graphs [6].

Lemma 2.2 *A p-doughnut graph always has a doughnut embedding.*

Let Γ be a doughnut embedding of a doughnut graph G. Let z_1, z_2, ..., z_{2p} be the vertices on C_2 in a clockwise order such that z_1 has exactly one neighbor on C_1 and exactly two neighbors on C_3. Let x_1, x_2, ..., x_p be the vertices on C_1 in a clockwise order where x_1 is the neighbor of z_1. Let y_1, y_2, ..., y_p be the vertices on C_3 in a clockwise order such that y_1 and y_2 are the neighbors of z_1. In the rest of the paper for any doughnut embedding of G, we follow the labeling of vertices on cycles C_1, C_2 and C_3 as mentioned above. We now have the following lemmas from [6].

Lemma 2.3 *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let z_i be a vertex of C_2. Then the following conditions hold.*

(a) z_i has exactly two neighbors on C_1 and exactly one neighbor on C_3 if i is even. The neighbors of z_i on C_1 are x_p and x_1 if $i = 2p$ otherwise $x_{i/2}$ and $x_{i/2+1}$, and the neighbor of z_i on C_3 is y_1 if $i = 2p$ otherwise $y_{i/2+1}$.

(b) z_i has exactly two neighbors on C_3 and exactly one neighbor on C_1 if i is odd. The neighbors of z_i on C_3 are y_p and y_1 if $i = 2p - 1$ otherwise $y_{\lceil i/2 \rceil}$ and $y_{\lceil i/2 \rceil + 1}$, and the neighbor of z_i on C_1 is $x_{\lceil i/2 \rceil}$.

Lemma 2.4 *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let x_i be a vertex of C_1. Then x_i has exactly three neighbors z_{2p}, z_1, z_2 if $i = 1$ otherwise z_{2i-2}, z_{2i-1}, z_{2i} on C_2 in a clockwise order.*

Lemma 2.5 *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let y_i be a vertex of C_3. Then y_i has exactly three neighbors z_{2p-1}, z_{2p}, z_1 if $i = 1$ otherwise z_{2i-3}, z_{2i-2}, z_{2i-1} on C_2 in a clockwise order.*

A *Hamiltonian cycle (path)* of a graph G is a cycle (path) which contains all the vertices of G. We call a graph G *Hamiltonian* if G contains a Hamiltonian cycle. The Hamiltonian cycle problem asks whether a given graph contains a Hamiltonian cycle, and the problem is NP-complete even for 3-connected planar graphs [3]. However the problem becomes polynomial-time solvable for 4-connected planar graphs: Tutte proved that a 4-connected planar graph necessarily contains a Hamiltonian cycle [14]. We call a graph G is *Hamiltonian-connected* if G has a Hamiltonian path between any pair of vertices of G. Thomassen proved that 4-connected planar graphs are Hamiltonian-connected [13].

3 Finding Hamiltonian Path in Doughnut Graphs

A doughnut graph G is Hamiltonian-connected since G is 5-connected. One can find a Hamiltonian path in a doughnut graph using algorithm proposed by Chiba and Nishizeki [1]. In their paper, they gave a proof of Tutte's theorem based on Thomassen's short proof avoiding decomposition of a 4-connected planar graph into nondisjoint subgraphs. Their proof is constructive and yields an algorithm for finding Hamiltonian path. Their algorithm clearly runs in $O(n^2)$ time, since one step of divide-and-conquer can be done in $O(n)$ time. The key idea for linear implementation of this algorithm is to use, in place of the Hopcroft and Tarjan's algorithm [5], a new algorithm to decompose a plane graph into small subgraphs by traversing some facial cycles. Although a sophisticated analysis shows that each of the edge is traversed at most constant time during one execution of Hamiltonian path finding algorithm and hence the algorithm runs in linear time, the linear-time implementation of the algorithm looks non-trivial. In this section we present a very simple linear-time algorithm for finding Hamiltonian path between any pair of vertices of a doughnut graph. In our algorithm we exploit the simple structure of a doughnut graph.

We have the following theorem on a doughnut graph.

Theorem 3.1 *Let G be a doughnut graph. Then a Hamiltonian path between any pair of vertices of G can be found in linear time.*

Proof. We first show a Hamiltonian path between any pair of vertices of a doughnut graph. Let Γ be a doughnut embedding of G and let C_1, C_2 and C_3 be the outer cycle, the middle cycle and the inner cycle of Γ. We have the following four cases to consider.

Case 1: Both the vertices u, v are either on C_1 or on C_3.

We assume that both of u and v are on C_1, since the case where both of u and v are on C_3 is similar. Let $u = x_i$ and $v = x_j$. Without loss of generality, we assume that $i < j$. We take the following paths. (i) $P_1 = x_i$,

z_{2i-1}, z_{2i}, x_{i+1}, z_{2i+1}, z_{2i+2}, $\ldots$, x_{j-1}, z_{2j-3}, z_{2j-2}; (ii) $P_2 = y_j$, y_{j-1}, $\ldots$, y_{j+1}; (iii) $P_3 = z_{2j-1}$, z_{2j}, $\ldots$, z_{2i-2}; and (iv) $P_4 = x_{i-1}$, x_{i-2}, $\ldots$, x_j. The path P_1 contains vertices of C_1 and C_2. By Lemma 2.4, z_{2i-1} is a neighbor of x_i. By Lemma 2.3, x_{i+1} is a neighbor of z_{2i} since $2i$ is even. The path P_2 contains all the vertices of C_3. The path P_3 contains all the vertices of C_2 those are not appear in P_1 and the Path P_4 contains vertices of C_1 those are not appear in the path P_1. We can concatenate the paths P_1 and P_2 since y_j is a neighbor of z_{2j-2} by Lemma 2.3. The paths P_2 and P_3 can be concatenated since z_{2j-1} is a neighbor of y_{j+1} by Lemma 2.5. The paths P_3 and P_4 can also be concatenated since x_{i-1} is a neighbor of z_{2i-2} by Lemma 2.3. Thus we can concatenate the four paths and the resulting path is HP_{x_i,x_j} where $HP_{x_i,x_j} = P_1\ P_2\ P_3\ P_4$. The path HP_{x_i,x_j} is a Hamiltonian path since P_1, P_2, P_3 and P_4 contain all the vertices of G. Figure 4 illustrates the case where $u = x_2$ and $v = x_5$. In this example (i) $P_1 = x_2$, z_3, z_4, x_3, z_5, z_6, x_4, z_7, z_8; (ii) $P_2 = y_5, y_4, y_3, y_2, y_1$; (iii) $P_3 = z_9$, z_{10}, z_1, z_2; and (iv) $P_4 = x_1$, x_5. The Hamiltonian path is $HP_{x_2,x_5} = P_1P_2P_3P_4$.

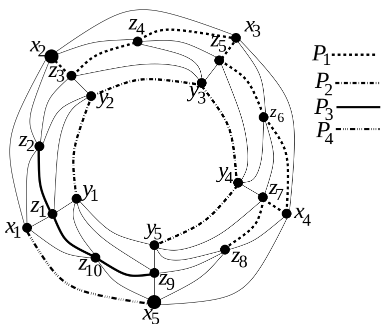

Figure 4: Illustration for case 1.

Case 2: Both vertices u, v are on C_2.

Let $u = z_i$ and $v = z_j$. Without loss of generality we may assume that $i < j$. We have two subcases to consider.

Subcase 2a: i is odd.

We take the following paths. (i) $P_1 = z_i$, $x_{\lceil i/2 \rceil}$, $x_{\lceil i/2 \rceil+1}$, $\ldots$, $x_{\lceil i/2 \rceil-1}$; (ii) $P_2 = z_{i-1}$, $y_{(i-1)/2+1}$, z_{i-2}, z_{i-3}, $y_{(i-3)/2}$, $\ldots$, $y_{(j+1)/2+1}$ if j is odd, otherwise $P_2 = z_{i-1}$, $y_{(i-1)/2+1}$, z_{i-2}, z_{i-3}, $y_{(i-3)/2}$, $\ldots$, z_{j+1}; (iii) $P_3 = y_{(j+1)/2}$, $y_{(j+1)/2-1}$, $\ldots$, $y_{\lceil i/2 \rceil+1}$ if j is odd, otherwise $P_3 = y_{\lceil (j+1)/2 \rceil}$, $y_{\lceil (j+1)/2 \rceil-1}$, $\ldots$, $y_{\lceil i/2 \rceil+1}$; and (iv) $P_4 = z_{i+1}$, z_{i+2}, $\ldots$, z_j. By Lemmas 2.3, 2.4, 2.5, we can prove the adjacency between two consecutive vertices of each path. We can also prove the adjacency between the end vertex and the starting vertex of paths P_1 and P_2, P_2 and P_3, P_3 and P_4 using the Lemmas 2.3, 2.4, 2.5. Therefore we can concatenate the paths P_1, P_2, P_3, P_4; and the resulting path $HP_{x_i,x_j} = P_1\ P_2\ P_3\ P_4$ is a Hamiltonian path since the paths P_1, P_2, P_3, P_4 contain all the vertices of the graph.

Subcase 2b: i is even.

We take the following paths. (i) $P_1 = z_i$, $x_{i/2+1}$, $x_{i/2+2}$, $\ldots$, $x_{i/2}$; (ii) $P_2 = z_{i-1}$, z_{i-2}, $y_{(i-2)/2+1}$, z_{i-3}, $\ldots$, $y_{(j+1)/2+1}$ if j is odd, otherwise $P_2 = z_{i-1}$, z_{i-2}, $y_{(i-2)/2+1}$, z_{i-3}, $\ldots$, z_{j+1}; (iii) $P_3 = y_{(j+1)/2}$, $y_{(j+1)/2-1}$, $\ldots$, $y_{\lceil i/2 \rceil+1}$ if j is odd, otherwise $P_3 = y_{\lceil (j+1)/2 \rceil}$, $\ldots$, $y_{\lceil i/2 \rceil+1}$; and (iv) $P_4 = z_{i+1}$, z_{i+2}, $\ldots$, z_j. Using the same arguments as in *Subcase* 2a, we can prove that $HP_{x_i,x_j} = P_1P_2P_3P_4$ is a Hamiltonian path.

Case 3: The vertex u either on C_1 or on C_3 and v on C_2.

We assume that u is on C_1 and v is on C_2, since the case where u is on C_3 and v is on C_2 is similar. Let $u = x_i$ and $v = z_j$. In this case, we have two subcases to consider.

Subcase 3a: j is even.

We take the following paths. (i) $P_1 = x_i$, x_{i+1}, $\ldots$, x_{i-1}; (ii) $P_2 = z_{2i-3}$, z_{2i-2}, y_i, z_{2i-1}, $\ldots$, z_{j-1}; (iii) $P_3 = y_{\lceil (j-1)/2 \rceil}$, $y_{\lceil (j-1)/2 \rceil} + 1$, $\ldots$, y_{i-1}; and (iv) $P_4 = z_{2i-4}$, z_{2i-5}, $\ldots$, z_j. Using the same arguments as in *Subcase* 2a, we can prove that $HP_{x_i,x_j} = P_1\ P_2\ P_3\ P_4$ is a Hamiltonian path.

Subcase 3b: j is odd.

We take the following paths. (i) $P_1 = x_i$, x_{i+1}, $\ldots$, x_{i-1}; (ii) $P_2 = z_{2i-3}, z_{2i-2}, y_i, z_{2i-1}, \ldots, y_{\lceil j/2 \rceil}$; (iii) $P_3 = y_{\lceil j/2 \rceil+1}, y_{\lceil j/2 \rceil+2}, \ldots, y_{i-1}$; and (iv) $P_4 = z_{2i-4}$, z_{2i-5}, $\ldots$, z_j. Using the same arguments as in *Subcase* 2a, we can prove that $HP_{x_i,x_j} = P_1\ P_2\ P_3\ P_4$ is a Hamiltonian path.

Case 4: The vertex u on C_1 and v on C_3.

We assume that u is on C_1 and v is on C_3 since the case where u is on C_3 and v is on C_1 is similar. Let us assume that $u = x_i$ and $v = y_j$. We take the following paths. (i) $P_1 = x_i$, x_{i+1}, $\ldots$, x_{i-1}; (ii) $P_2 = z_{2i-3}$, $y_{\lceil (2i-3)/2 \rceil}$, z_{2i-4}, z_{2i-5}, $y_{\lceil (2i-5)/2}$, $\ldots$, z_{2j-1}; (iii) $P_3 = z_{2j-2}$, z_{2j-3}, $\ldots$, z_{2i-2}; and (iv) $P_4 = y_i, y_{i+1}, \ldots, y_j$. Using the same arguments as in *Subcase* 2a, we can prove that $HP_{x_i,x_j} = P_1\ P_2\ P_3\ P_4$ is a Hamiltonian path.

Therefore G has a Hamiltonian path between any pair of vertices. One can find such a path in linear time easily. $\mathcal{Q.E.D.}$

4 k-Partition of a Doughnut Graph

A p-doughnut graph is a 5-connected planar graph. One may think that a p-doughnut graph G for $p \geq 5$ can be partitioned using Nagai and Nakano's [9] algorithm after triangulation of one of the face of G with p-vertices. But it is not possible since after removing the dummy edges used for triangulation the partition may not be connected. In this section, we give an algorithm for finding a k-partition of a doughnut graph. We have the following theorem.

Theorem 4.1 *Let G be a doughnut graph. Then G admits k-partitioning. Furthermore, one can find such a partition in linear time.*

Proof. By Theorem 3.1, G has a Hamiltonian path between any pair of vertices. We first find a Hamiltonian path $HP_{u,v}$ between any pair of vertices u and v of G. Then starting from one end vertex of $HP_{u,v}$, we divide the path into k subpaths where each subpath contains the number of vertices exactly equal to the natural number associated with the corresponding partition. Each of the partition is a subgraph induced by the vertices of the corresponding subpaths. Figure 5 illustrates a k-partitioning of G. Figure 5(i) illustrates a Hamiltonian path of G between vertices x_2 and z_6. Figure 5(ii) illustrates a k-partition of G for k = 7 where the natural numbers are 3, 2, 5, 3, 2, 4, 1, respectively. Figure 5(iii) illustrates a k-partition of G for $k = 4$ where the natural numbers are 4, 6, 3, 7, respectively. The edges of Hamiltonian path and the connected subgraphs are drawn by thick lines, and the remaining edges are drawn by thin lines. One can find a Hamiltonian path by Theorem 3.1 in linear time and a subgraph induced by the vertices on a subpath can also be obtained in linear time.

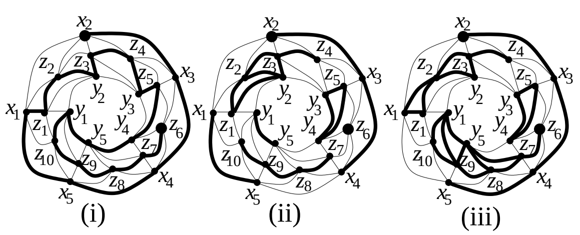

Figure 5: (i) Hamiltonian path HP_{x_2,z_6} of G, (ii) a 7-partition of G, and (iii) a 4-partition of G.

$\mathcal{Q.E.D.}$

Our k-partition algorithm is based on finding a Hamiltonian path between any pair of vertices of a doughnut graph. The two end vertices of a Hamiltonian path can be used as two basis vertices of a k-partition. So, the following theorem also holds.

Theorem 4.2 *Let G be a doughnut graph. Then G admits k-partitioning with basis at most two.*

By using the Chiba and Nishizeki's [1] algorithm, we now have the following result for any 4-connected planar graph.

Theorem 4.3 *Let G be a 4-connected planar graph. Then G admits k-partitioning with the basis at most two.*

5 Conclusion

In this paper, we gave a linear-time algorithm for finding a k-partition of a doughnut graph. A doughnut graph G is a fault tolerant graph since the vertices of G lies on three vertex disjoint cycles and G is 5-regular. Therefore k-partitioning of G is interesting. We can also have a k-partition for a 4-connected planar graph using the same method. Finding a 5-partition of a doughnut graph with basis five is left as an open problem.

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