Pseudo-stable Models for Logic Programs

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Abstract

For a general logic program, a set of $n+1$ logic values is considered and an undefined value denoted u. Partial multi-valued interpretations are also defined. A general logic program P may contain the constants that are defined for every logic value. A pseudo-negation denoted $\rceil_h$ is defined for every integer h, where $0 \leq h < n$. A partial ordering denoted $\preceq_h$ is defined between multi-valued interpretations. Using an operator ψ defined for a program P, a multi-valued pseudo-stable semantics for the program P is introduced. The pseudo-stable models which satisfy certain properties are minimal elements of the set of all models for the program P having those properties. The class of pseudo-stable models for a program P contains strictly the class of 3-valued stable models for P.
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1 Introduction

The stable model semantics has been introduced by M. Gelfond and V. Lifschitz in [6] and by N. Bidoit and C. Froidevaux in [1]. This semantics is strongly connected to well-founded semantics. The well-founded semantics has been introduced by A. Van Gelder, K. Ross and J. Schlipf in [14]. It is a 3-valued semantics. They use the logical values: $true$, $false$ and $\perp$ (an unknown truth value). They have shown that if a logic program P has a 2-valued well-founded model, then this model is the unique stable model of P. T. Przymusinski in [11] has introduced 3-valued stable models as a generalization of 2-valued stable models. He has shown that the well-founded model of any program P coincides with the smallest 3-valued stable model of P. M. Fitting in [5] has studied the structure of the family of all stable models for a logic program using two orderings: one is called the knowledge ordering based on degree of definedness, the second is called the truth ordering based on degree of truth. He shows that in the first ordering, every logic program has a smallest stable model, which coincides with the well-founded model. T. Przymusinski in [12] has introduced the stable model semantics for disjunctive logic programs and deductive databases. He shows that for normal programs, the partial disjunctive stable semantics coincides with the well-founded semantics. E. Dantsin, T. Eiter, G. Gottlob and A. Voronkov have shown in [2] that the stable models are more expressive than the well-founded models. M. Gelfond and V. Lifschitz extended the stable model semantics to programs with the classical negation [7]. This semantics is called the answer set semantics. The same authors extended this semantics to programs with disjunction in [8]. V. Lifschitz, L. R. Tang and H. Turner in [9] extended the answer set semantics to programs with nested expressions. In [3] the author defined a semantics of type well-founded considering the set of logic values as a complete lattice. In [4] we defined a stable semantics which extends the stable semantics defined by T. Przymusinski in [12].
Mathematical foundations about interpretations, models, monotonic operators, fixed-points and other notions concerning logic programs are found in [10].
In this paper we propose a pseudo-stable semantics for a program P, considering the pseudo-negations, partial orderings between interpretations and monotonic operators with respect to partial orderings.

2 Motivation

In logic programs the negation operator $\sim$ is used to represent negation as failure. In the case of n-valued semantics of logic programs, if I is an interpretation, then the values for I are considered in the set $\{0, 1/n, \ldots, 1\}$, where 0 is $false$, 1 is $true$ and k/n are intermediate logic values between $false$ and $true$, $1 \leq k < n$. The function I is extended to $\widehat{I}$ for all closed formulae of a certain language L. In particular, the following relations are defined: $\widehat{I}(A)$=$I(A)$, for every ground atom A and $\widehat{I}(\sim S)$= $1 - \widehat{I}(S)$, for every closed formula S. That means the operator $\sim$ is idempotent. There are many situations, for which the operator $\sim$ is not idempotent. To illustrate this point, let us consider the following program:

$$issickleave(X) \quad \leftarrow \sim isgoodhealth(X), \sim haspermission(X)$$

$$isgoodhealth(a) \quad \leftarrow c_{p/n}$$

$$haspermission(a) \leftarrow c_{q/n}$$

In this program the predicate $issickleave(X)$ represents the property of the person X who must sick leave, the predicate $isgoodhealth(X)$ specifies the level of the health for the person X and the predicate $haspermission(X)$ represents a permission degree(from his doctor) for the person X to work. The constant a represents a person. $c_{p/n}$ and $c_{q/n}$ are constant propositions coresponding to logic values p/n, q/n, respectively. The semantics for the first rule is:"if a person has not a good health and he has not a permission to work from his doctor, then he must sick leave". The second rule has the semantics:"the health level must be at least p/n". A similar semantics is considered for the third rule. In this program we are interested in models for which the predicate $issickleave$ has only logic values $true$ or $false$. Moreover, we make the following assumption:"if the goodhealth level of a person is at most h/n and he has the permission to work(from his doctor) at most the level h/n, then he must sick leave". A solution for such programs is to consider the interpretation of the negation $\sim$ in the following manner:

a) $\rceil_h v = 1$, for every v in $L_n, v \leq h/n$.

b) $\rceil_h v = 0$, for every v in $L_n, v > h/n$.

This function denoted $\rceil_h$ is called a pseudo-negation.

3 Multi-valued Interpretations

We consider a clause having the form $A \leftarrow B_1, \ldots, B_m, \sim D_1, \ldots, \sim D_q$, where $A, B_j, D_i, 1 \leq j \leq m, 1 \leq i \leq q$, are atoms. A program P is a finite set of clauses. Let us define multi-valued interpretations. Let H be the Herbrand base constructed for a program P. We consider the following set of truth values: $L_n = \{0, 1/n, \ldots, (n-1)/n, 1\}$. Let h be an integer, such that $0 \leq h < n$. The value 0 from L_n corresponds to "false", the value 1 corresponds to "true" and k/n is an intermediate value between "false" and "true", for every $k, 1 \leq k < n$. Moreover, we use an undefined value denoted by u. The intuitive meaning of the undefined truth value u is possible, rather than a truth value from L_n. Therefore, the undefined status of an atom A in a given model M of a theory T indicates that M assigns some, but only unknown truth value to A.

For every v from L_n, let us define by c_v the proposition which has the true value v, for every ground atom of H. c_v is called a constant and our positive logic programs are allowed to contain propositions c_v among their premises. Let us denote by c_u the constant proposition corresponding to the undefined value u.

Definition 1 *A multi-valued Herbrand interpretation I on L_n is a $n+1$-uple $(F_0^I, \ldots, F_n^I)$, where F_i^I are disjoint subsets of the Herbrand base $H, 0 \leq i \leq n$. Let us consider the subset F_u^I defined as the set of all remaining atoms in $H - \bigcup_{i=0}^{n} F_i^I$. The interpretation extended to H corresponding to I will be represented as $(F_0^I, \ldots, F_n^I, F_u^I)$.*

In the following we consider only Herbrand interpretations and models. An interpretation $I = (F_0^I, \ldots, F_n^I)$ can be viewed as a partial function $I : H \rightarrow L_n$ defined as follows: $I(A) = k/n$ iff $A \in F_k^I$ for every $0 \leq k \leq n$. In the case $A \in F_u^I$ we say that $I(A)$ is undefined and we write $I(A) = u$. Thus, every interpretation I is considered as a total function $I : H \rightarrow L_n \cup \{u\}$. We consider a pseudo-negation associated with the integer h, denoted by $\rceil_h$ and having the following properties:

a) $\alpha < \beta$ implies $\rceil_h \beta \leq \rceil_h \alpha$, for every $\alpha, \beta \in L_n$.

b) $\alpha \leq h/n$ implies $\rceil_h \alpha > h/n$ for every $\alpha \in L_n$.

c) $\alpha > h/n$ implies $\rceil_h \alpha \leq h/n$ for every $\alpha \in L_n$.

d) $\rceil_h u = u$

This pseudo-negation differs from a negation because the property of idempotence for the pseudo-negation is not satisfied. The relation " $<$ " is the totally ordered relation from L_n. The relation $\alpha \leq \beta$ means $(\alpha < \beta)$ or $(\alpha = \beta)$.
For $n = 1$ and $h = 0$ we consider $\rceil_0$ defined as: $\rceil_0 0 = 1$, $\rceil_0 1 = 0$ (it coincides with the negation used by T. Przymusinski in [11]).
The interpretation I considered as a total function

on H can be extended recursively using the pseudo-negation $\rceil_h$ to the truth valuation $\widehat{I}_h$ from C to $L_n \cup \{u\}$, where the set C consists of all literals of H and all conjunctions of literals:

$\widehat{I}_h(A) = I(A)$, for every ground atom A,

$\widehat{I}_h(\sim A) = \rceil_h I(A)$, for every ground atom A, such that $I(A) \neq u$,

$\widehat{I}_h(\sim A) = 1$, for every ground atom A, such that $I(A) = u$,

$\widehat{I}_h(L_1 \wedge \ldots \wedge L_p) = \min_{1 \leq i \leq p} \{\widehat{I}_h(L_i), \widehat{I}_h(L_i) \neq u\}$.

We must consider $min\ \phi = 1$, where ϕ is the empty set.

That means the negation $\sim$ from the clauses bodies of a program P is interpreted via the pseudo-negation $\rceil_h$.

Definition 2 *An interpretation I satisfies the ground instantiated rule r of P (or I is a model for r), having the form: $r \equiv A \leftarrow L_1, \ldots, L_p$ iff:*

a) $\widehat{I}_h(A) = u$ or

b) $\widehat{I}_h(A) \neq u$ and $\widehat{I}_h(L_1 \wedge \ldots \wedge L_p) \leq \widehat{I}_h(A)$.

Definition 3 *An interpretation I is a model for the program P iff I is a model for every ground instantiated rule r of P.*

In the following we define a partial ordering between multi-valued interpretations, associated to the integer h and denoted by $\preceq_h$, where $0 \leq h < n$.

Definition 4 *Let $I = (F_0^I, \ldots, F_n^I)$ and $J = (F_0^I, \ldots, F_n^I)$ be two interpretations. We say that $I \preceq_h J$ if the following inclusions are satisfied:*

a) $F_j^J \subseteq F_0^I \cup \ldots \cup F_j^I$, for every $j, 0 \leq j \leq h$ and

b) $F_j^I \subseteq F_j^J \cup \ldots \cup F_n^J$ for every $j, h+1 \leq j \leq n$.

For an integer h fixed, minimal and least models of a program P with respect to $\preceq_h$ ordering, minimize the degree of truth of atoms, by maximizing the sets of atoms that correspond to truth values $j/n, 0 \leq j \leq h$ and by minimizing the sets of atoms that correspond to truth values $j/n,\ h < j \leq n$.

Remark 1 *In case $n = 1$ and $h = 0$ the ordering $\preceq_h$ coincides with the standard ordering $\preceq$ used by T.Przymusinski in [11] to study three-valued stable models.*

Proposition 1 *Let I and J be two interpretations, such that $I \preceq_h J$. We have:*

i) $F_u^J \subseteq F_0^I \cup \ldots \cup F_h^I \cup F_u^I$

ii) $F_u^I \subseteq F_u^J \cup F_{h+1}^J \cup \ldots \cup F_n^J$

iii) $F_j^J \subseteq F_0^I \cup \ldots \cup F_j^I \cup F_u^I$, for every $j, h+1 \leq j \leq n$.

iv) $F_j^I \subseteq F_u^J \cup F_j^J \cup \ldots \cup F_n^J$, for every $j, 0 \leq j \leq h$.

Proof

(i) Let A be an atom from F_u^J. Assume that $A \notin F_0^I \cup \ldots \cup F_h^I \cup F_u^I$. It results that $A \in F_{h+1}^I \cup \ldots \cup F_n^I$. Using the part b) from Definition 4, we have $A \in F_{h+1}^J \cup \ldots \cup F_n^J$, which is impossible. The proofs of (ii)-(iv) are similar to that of (i).

4 Multi-valued Pseudo-stable Models

In this section we define multi-valued pseudo-stable models. Firstly, we define an operator, denoted ψ, on the set of all multi-valued interpretations of a program P.

Definition 5 *Let P be a logic program and I be a multi-valued interpretation of P. Let $\psi(I) = (F_0^{\psi(I)}, \ldots, F_n^{\psi(I)})$ be the interpretation defined by the following relations:*

a) For every $j, 0 \leq j \leq h$ and A a ground atom, we have: $A \in F_j^{\psi(I)}$ iff the conditions a1) and a2) are satisfied.

a1) for every ground instantiated rule r of P, having the form $r \equiv A \leftarrow L_1, \ldots, L_m$, we have $\widehat{I}_h(L_1 \wedge \ldots \wedge L_m) \leq j/n$.

a2) there exists a ground instantiated rule r_1 of P of the form $r_1 \equiv A \leftarrow Q_1, \ldots, Q_p$, such that $\widehat{I}_h(Q \wedge \ldots \wedge Q_p) = j/n$.

b) For every $j, h+1 \leq j < n$ (in the case $h+1 < n$) and A a ground atom, we have: $A \in F_j^{\psi(I)}$ iff the conditions b1) and b2) are fulfilled:

b1) for every ground instantiated rule r_3 of P, having the form $r_3 \equiv A \leftarrow W_1, \ldots, W_s$, we have: $\widehat{I}_h(W_i) \neq u$ for every $i, 1 \leq i \leq s$, $\widehat{I}_h(W_1 \wedge \ldots \wedge W_s) \leq j/n$ and $\widehat{I}_h(W_i) \geq (h+1)/n$ for every $i, 1 \leq i \leq s$.

b2) there exists a ground instantiated rule r_2 of P having the form $r_2 \equiv A \leftarrow V_1, \ldots, V_q$, such that:

$$\widehat{I}_h(V_1 \wedge \ldots \wedge V_q) = j/n$$

c) A ground atom A is in $F_n^{\psi(I)}$ if there exists a ground instantiated rule r of P having the form $A \leftarrow L_1, \ldots, L_m$ such that $\widehat{I}_h(L_i) = 1$ for every $i, 1 \leq i \leq m$.

Remark 2 *A ground atom A belongs to $F_0^{\psi(I)}$ iff for every instantiated rule r of P, having the form: $r \equiv A \leftarrow L_1, \ldots, L_m$, there is $i, 1 \leq i \leq m$, such that $\widehat{I}_h(L_i) = 0$.*

Remark 3 *In case $n = 1$ and $h = 0$ the operator ψ from Definition 5 coincides with the operator ψ defined by T.Przymusinski in [11].*

The following theorem shows the monotonic property of the operator ψ with respect to the partial ordering $\preceq_h$, for positive programs.

Theorem 1 *Let P be a positive program and ψ be the operator as it was considered in Definition 5. The operator ψ is monotonic with respect to the ordering $\preceq_h$ between multi-valued interpretations.*

Proof Let $I \preceq_h J$. Since $F_0^J \subseteq F_0^I$, using the Remark 1, the relation $F_0^{\psi(J)} \subseteq F_0^{\psi(I)}$ follows immediately.

Let j be a positive integer , such that $1 \leq j \leq h$ and $A \in F_j^{\psi(J)}$. Then we have: for every ground instantiated rule r of P of the form:

$$r \equiv A \leftarrow L_1, \ldots, L_m \text{ ,it results that}$$

$$\widehat{J}_h(L_1 \wedge \ldots \wedge L_m) \leq j/n \tag{1}$$

and there is a ground instantiated rule r_1 of P having the form:

$$r_1 \equiv A \leftarrow Q_1, \ldots Q_s, \text{ such that}$$

$$\widehat{J}_h(Q_1 \wedge \ldots \wedge Q_s) = j/n \tag{2}$$

From the relation (1) we obtain that there is a natural number $p, 1 \leq p \leq m$, such that:

$$\widehat{J}_h(L_p) \leq j/n \tag{3}$$

Since $j \leq h$ and L_p is an atom, we obtain $\widehat{I}_h(L_p) \leq j/n$ which implies

$$\widehat{I}_h(L_1 \wedge \ldots \wedge L_m) \leq j/n \tag{4}$$

Let us denote by M_A the set of all ground instantiated rule of P, whose head is A. If $r \in M_A$ has the form $A \leftarrow L_1, \ldots, L_m$, we denote by $v_{A,r}^I$ the following logic value: $\widehat{I}_h(L_1 \wedge \ldots \wedge L_m)$.

Let $v_A^I = \max_{r \in M_A} \{v_{A,r}^I\}$. From the relation (4) we obtain:

$$v_{A,r}^I \leq j/n \text{ and } v_A^I \leq j/n \tag{5}$$

Moreover, there exists r_1 from M_A, such that:

$$v_{A,r_1} = v_A^I \tag{6}$$

If we denote the truth value v_A^I by q/n, it results that $0 \leq q \leq j$, therefore $A \in F_q^{\psi(I)}$. We have shown that

$$F_j^{\psi(J)} \subseteq F_0^{\psi(I)} \cup \ldots \cup F_j^{\psi(I)}, 1 \leq j \leq h \tag{7}$$

Now, let j be an integer, such that $h+1 \leq j < n$ (in the case $h+1 < n$) and A be an atom from $F_j^{\psi(I)}$. The conditions b1) and b2) from Definition 5 are fulfilled.

For every ground instantiated rule r_3 of P, having the form $r_3 \equiv A \leftarrow W_1, \ldots, W_s$, we have

$$\widehat{I}_h(W_i) \geq (h+1)/n, \text{ for every } i, 1 \leq i \leq s \tag{8}$$

Since W_i are atoms and using the relation $I \preceq_h J$, we obtain

$$\widehat{J}_h(W_i) \geq (h+1)/n, \text{ for every } i, 1 \leq i \leq s \tag{9}$$

which implies:

$$\widehat{J}_h(W_1 \wedge \ldots \wedge W_s) \geq (h+1)/n \tag{10}$$

Let us denote by $M_{A,d}$ the set of all ground instantiated rule of P, whose head is A and every atom from the body is defined for the interpretation J. Let us define v_{A,r_3}^J by following logic value:

$$v_{A,r_3}^J = \widehat{J}_h(W_1 \wedge \ldots \wedge W_s) \tag{11}$$

Let k be the integer, such that

$$k/n = \max_{r_3 \in M_{A,d}} \{v_{A,r_3}^J\} \tag{12}$$

Since b2) is satisfied, it results that

$$k \geq j \tag{13}$$

The relations (9),(11) and (12) imply:

$$A \in F_k^{\psi(J)} \tag{14}$$

We have shown that for every j, such that $h+1 \leq j < n$ (when $h+1 < n$), we have:

$$F_j^{\psi(I)} \subseteq F_j^{\psi(J)} \cup \ldots \cup F_n^{\psi(J)} \tag{15}$$

Using the assertion c) and $I \preceq_h J$, we obtain

$$F_n^{\psi(I)} \subseteq F_n^{\psi(J)} \tag{16}$$

Thus, the relations (7), (15) and (16) are equivalent with $\psi(I) \preceq_h \psi(J)$.

The following theorem establishes the existence of the least model for a positive program P and a monotonic operator with respect to the ordering $\preceq_h$.

Definition 6 *Let η be an operator defined on the set of all multi-valued interpretations of a program P and I be an interpretation of a program P. The operator η is said constrained for I with respect to the ordering $\preceq_h$ iff $\eta(I) \preceq_h I$.*

Definition 7 *We say that an interpretation I has the property (P_1) iff $I(A) \neq u$ for every atom A which is the head of a ground instantiated rule of P.*

We said that an interpretation I has the property (P_2) iff $I(C) \neq u$ for every ground atom C, such that $\sim C$ appears in the body of a ground instantiated rule of P.

Proposition 2 *Let ψ be the operator as it was defined in Definition 5 and M be a model for the program P with the property (P_1). Then ψ is constrained for M with respect to $\preceq_h$.*

Proof Let $\psi(M) = (F_0^{\psi(M)}, \ldots, F_n^{\psi(M)})$, where the sets of atoms $F_j^{\psi(M)}$ are specified in Definition 5 for every j, $0 \leq j \leq h$.

Firstly, we must show that:

$$F_j^M \subseteq F_0^{\psi(M)} \cup \ldots \cup F_j^{\psi(M)},$$

$$\text{for every } j,\ 0 \leq j \leq h \tag{17}$$

where $M = (F_0^M, \ldots, F_n^M)$.

Let A be an atom from $F_j^M, 0 \leq j \leq h$. This means $M(A) = j/n$.

Since M is a model for P, it results: for every $A \leftarrow L_1, \ldots, L_m$ a ground instantiated rule of P, whose head is A, we have:

$$\widehat{M}_h(L_1 \wedge \ldots \wedge L_m) \leq M(A) = j/n$$

Hence, there is an integer i, such that $0 \leq i \leq j$ and $A \in F_i^{\psi(M)}$, therefore the relations (17) are true.

Secondly, we show that:

$$F_j^{\psi(M)} \subseteq M_j \cup \ldots \cup M_n,$$

$$\text{for every } j,\ h+1 \leq j \leq n \tag{18}$$

Let A be an atom from $F_j^{\psi(M)}$.

There exists a ground instantiated rule r_2 of P, having the form $r_2 \equiv A \leftarrow V_1, \ldots, V_l$, such that $\widehat{M}_h(V_i) \neq u$ for every $i, 1 \leq i \leq l$ and $\widehat{M}_h(V_1 \wedge \ldots \wedge V_l) = j/n$. Since $M(A) \neq u$ and M is a model for r_2, it results $M(A) \geq j/n$, hence $A \in M_j \cup \ldots \cup M_n$, therefore (18) is true.

Definition 8 *Let I be an interpretation and $\mathcal{M}$ a set of interpretations such that $I \in \mathcal{M}$. We say that I is the least element of $\mathcal{M}$ with respect to $\preceq_h$ iff $I \preceq_h J$, for every $J \in \mathcal{M}$. This element I will be denoted $\mathcal{M}$ - least interpretation. Besides this, if I is a model for P, it will be denoted $\mathcal{M}$ - least model of P. Let us denote by $\mathcal{M}_{P_1}$ the set of models of the program P with the property (P_1).*

Theorem 2 *Let P be a positive program and ψ be the operator as it was defined in Definition 5. There exists the least fixed point of the operator ψ, denoted $M(P,\psi)$ with respect to the ordering $\preceq_h$. Moreover, $M(P,\psi)$ is a model of P and if the model $M(P,\psi)$ has the property (P_1), then $M(P,\psi)$ is the $\mathcal{M}_{P_1}$ - least model of P.*

Proof The Theorem 1 emphasizes that the operator ψ is monotonic with respect to $\preceq_h$. Using Proposition 2 we have ψ is constrained for every model $M \in \mathcal{M}_{P_1}$ with respect to $\preceq_h$.

The model $M(P,\psi)$ can be obtained by iterating ω times the operator ψ on the interpretation $\perp$, where ω is the first ordinal and $\perp$ is the least interpretation with respect to $\preceq_h$, hence $\perp = (H, \phi, \ldots, \phi)$, that means $F_0^{\perp} = H$ and $F_j^{\perp} = \phi$ (the empty set), for every $j, 1 \leq j \leq n$. It follows that $M(P,\psi)$ is the least fixed point of the operator ψ.

Let us show that $M(P,\psi)$ is a model for P. Let us denote the model $M(P,\psi)$ by M. We have $\psi(M) = M$.

Using the representation of an interpretation as a vector of sets, we obtain: $(F_0^{\psi(M)}, \ldots, F_n^{\psi(M)}) = (F_0^M, \ldots, F_n^M)$, hence $F_j^{\psi(M)} = F_j^M$ for every $j, 0 \leq j \leq n$. These relations imply:

$$A \in F_j^{\psi(M)} \text{ iff } A \in F_j^M \text{ iff}$$
$$M(A) = j/n \text{ for every } j, 0 \le j \le n.$$

In the case $0 \le j < n$ using the definitions of $F_j^{\psi(M)}$, we have: for every ground instantiated rule r of P having the form $r \equiv A \leftarrow L_1, \dots, L_m$, $A \in F_j^{\psi(M)}$ implies $\widehat{M}_h(L_1 \wedge \dots \wedge L_m) \le j/n = \widehat{M}_h(A)$, hence M satisfies r.

In the case $j = n$ we have $A \in F_n^{\psi(M)}$ iff $A \in F_n^M$ iff $M(A) = 1$.

The relation $M(A) = 1$ implies that M satisfies every ground instantiated rule of P whose head is A.

If $A \in F_u^{\psi(M)}$, then $A \in F_u^M$, which implies M satisfies every ground instantiated rule r of P with the form $A \leftarrow L_1, \dots, L_m$.

Let us consider the case $M = M(P, \psi)$ having the property (P_1). Let M' be an arbitrary model from $\mathcal{M}_{P_1}$. Using proposition 2, we obtain the operator ψ is constrained for M' with respect to $\preceq_h$, hence $\psi(M') \preceq_h M'$.

Using the relations $\bot \preceq_h M'$, $\psi(M') \preceq_h M'$ and the monotonicity of the operator ψ with respect to $\preceq_h$, we obtain that $M(P, \psi) \preceq_h M'$, hence $M(P, \psi)$ is the $\mathcal{M}_{P_1}$ - least model.

The following example points out that, in general, the model $M(P, \psi)$ is not the $\mathcal{M}^P$ - least model of the program P, where $\mathcal{M}^P$ is the set of all models of P.

Example 1 *Let P_1 be the following program:*

$a \leftarrow b, \sim c, \sim d$

$a \leftarrow a, \sim d$

$b \leftarrow a, \sim c$

$c \leftarrow c_{2/3}$

$d \leftarrow c_{1/3}$

Let $h = 1, n = 3$ and the pseudo-negation $\rceil_1$, which is a negation, defined as: $\rceil 0 = 1, \rceil 1/3 = 2/3, \rceil 2/3 = 1/3, \rceil 1 = 0$.

The interpretation $M = (F_0^M, \dots, F_3^M)$, where $F_0^M = \{a, b\}$, $F_1^M = \{d\}, F_2^M = \{c\}, F_3^M = \phi$ is $M(P_1, \psi)$. If we consider the interpretation $M_2 = (F_0^{M_2}, \dots, F_3^{M_2})$, where $F_0^{M_2} = \{a, b\}$ and $F_i^{M_2} = \phi$, for every $i, 1 \le i \le 3$, then M_2 is a model for P_1, but $M \not\preceq_1 M_2$.

Now, we introduce a new operator on multi-valued interpretations, which depends on the operator ψ, the integer h and on pseudo-negation $\rceil_h$. It will be denoted by $\Gamma^*(\psi, h, \rceil_h)$. This operator is of type Gelfond-Lifschitz operator [6].

Definition 9 *Let P be a general logic program with negation, denoted $\sim$, let h be a positive integer such that $0 \le h < n$ and $\rceil_h$ a pseudo-negation defined on L_n. Let I be a multi-valued interpretation. We denote by P/I the program obtained from P by replacing in every ground instantiated clause of P all negative literals $L = \sim A$ by the constant c_v, where $v = \rceil_h I(A)$.*

The resulting program P/I is positive, hence by the Theorem 2, there is the least fixed point of ψ (a model for P/I), denoted $M(P/I, \psi)$. Let us define the operator $\Gamma^*(\psi, h, \rceil_h)$ by the following relation:

$$\Gamma^*(\psi, h, \rceil_h)(I) = M(P/I, \psi).$$

Remark 4 *If the constant c_1 belongs to the body of a clause, then it may be eliminated from this clause. If the constant c_0 belongs to the body of a clause, then this clause can be removed.*

Definition 10 *Every fixed point of $\Gamma^*(\psi, h, \rceil_h)$ is called a multi-valued pseudo-stable model for P with respect to the operator ψ, the ordering $\preceq_h$ and pseudo-negation $\rceil_h$.*

Remark 5 *Every 3-stable model is a multi-valued pseudo-stable model.*

Proof The proof results from Remarks 1 and 3.

5 Minimal Pseudo-stable Models

In this section we show that if a multi-valued pseudo-stable model has the properties (P_1) and (P_2), then it is a minimal model of P on the set of all models of P with the properties (P_1) and (P_2), with respect to the ordering $\preceq_h$.

Theorem 3 *Let M be a pseudo-stable model for P as it was defined in Definition 10. If M has both of properties (P_1) and (P_2), then M is a minimal element of $\mathcal{M}_{P_1} \cap \mathcal{M}_{P_2}$ (the set of all models of P which has the properties (P_1) and (P_2)), with respect to $\preceq_h$.*

Proof Let M be a fixed point of the operator $\Gamma^*(\psi, h, \rceil_h)$. Firstly, we show that M is a model for P. Then, it will result that M is a model for P/M (using Theorem 2).

Let us consider an arbitrary ground instantiated clause from P, having the form:

$$r \equiv A \leftarrow B_1, \dots, B_m, \sim D_1, \dots, \sim D_q \qquad (19)$$

The corresponding clause to r from P/M is r':

$$r' \equiv A \leftarrow B_1, \ldots, B_m, c_{v_1}, \ldots, c_{v_q} \quad (20)$$

where $v_j = \rceil_h M(D_j), j = \overline{1,q}$. In the case $M(D_j) = u$, we take $v_j = u$ and $c_u = u$. It results:

M is a model for r iff

$$M \text{ is a model for } r' \quad (21)$$

Since M is a model for P/M, it results that M is a model for P.

Secondly, let us consider M having the properties (P_1) and (P_2). This implies $M \in \mathcal{M}_{P_1} \cap \mathcal{M}_{P_2}$. We must show that M is a minimal element of $\mathcal{M}_{P_1} \cap \mathcal{M}_{P_2}$ with respect to $\preceq_h$.

Let M_1 be a model from $\mathcal{M}_{P_1} \cap \mathcal{M}_{P_2}$, such that $M_1 \preceq_h M$. We show that $M_1 = M$.

Let r and r' be the clauses from (1) and (2) respectively and r'' be the clause from P/M_1 with the form:

$$r'' \equiv A \leftarrow B_1, \ldots, B_m, c_{w_1}, \ldots, c_{w_q} \quad (22)$$

with $w_j = \rceil_h M_1(D_j), j = \overline{1,q}$. If $M_1(D_j) = u$ then $w_j = u$.

Since we have M_1 is a model for r iff M_1 is a model for r'' and M_1 is a model for P, it results that

$$M_1 \text{ is a model for } r'' \quad (23)$$

We show that M_1 is also a model for r' (24)

If $M_1(A) = u$, then the relation (24) is true. Let us consider the case $M_1(A) \neq u$.

For a ground instantiated rule r of P, having the form: $r \equiv A \leftarrow L_1, \ldots, L_p$, let us denote by $body(r)$ the conjunction: $(L_1 \wedge \ldots \wedge L_p)$. We define by $v^{M_1}_{A,r''}$ and $v^{M_1}_{A,r'}$, the following logical values:

$$v^{M_1}_{A,r''} = \widehat{M}_{1,h}(\text{body } (r'')) \quad (25)$$

$$v^{M_1}_{A,r'} = \widehat{M}_{1,h}(\text{body } (r')) \quad (26)$$

We have:

$$v^{M_1}_{A,r''} = \ min\ \{\widehat{M}_{1,h}(B_1 \wedge \ldots \wedge B_m), F\} \quad (27)$$

where $F = \min\limits_{1 \le j \le q} \{c_{w_j}, c_{w_j} \neq u\}$ and

$$v^{M_1}_{A,r'} = \ min\ \{\widehat{M}_{1,h}(B_1 \wedge \ldots \wedge B_m), G\} \quad (28)$$

where $G = \min\limits_{1 \le j \le q} \{c_{v_j}, c_{v_j} \neq u\}$. We intend to show that:

$$G \le F \quad (29)$$

Since M_1 is a model from $\mathcal{M}_{P_1} \cap \mathcal{M}_{P_2}$, we obtain $w_j \neq u$ for every $j, 1 \le j \le q$. Similarly, we have $v_j \neq u$ for every $j, 1 \le j \le q$.

The relation (29) becomes:

$$\min_{1 \le j \le q} \{c_{v_j}\} \le \min_{1 \le j \le q} \{c_{w_j}\} \quad (30)$$

Using the relation $M_1 \preceq_h M$, Definition 4 and the properties of the pseudo-negation $\rceil_h$, we obtain $v_j \le w_j$ for every $j, 1 \le j \le q$, which implies (30). From the relations (28), (27) and (30) it results:

$$v^{M_1}_{A,r'} \le v^{M_1}_{A,r''} \quad (31)$$

But M_1 is a model for r'', hence we have:

$$v^{M_1}_{A,r''} \le M_1(A) \quad (32)$$

From the relations (31) and (32) it results that M_1 is a model for r', hence the assertion (24) is true.

We have shown the following:

$$M_1 \text{ is a model for } P/M \text{ and } M_1 \in \mathcal{M}_{P_1} \cap \mathcal{M}_{P_2} \quad (33)$$

Using the results of Theorem 2, we obtain:

$$M \preceq_h M_1 \quad (34)$$

From (34) and $M_1 \preceq_h M$ it results $M = M_1$.

The following Remark points out that, in general, a pseudo-stable model for P is not minimal with respect to $\preceq_h$ and for all models of P.

Remark 6 *The interpretation M from Example 1 is a pseudo-stable model for P_1, but it is not minimal with respect to $\preceq_1$ and for all models of P_1.*

Proof The interpretation $M_3 = (F_0^{M_3}, \ldots, F_3^{M_3})$, where $F_0^{M_3} = \{a, b\}, F_1^{M_3} = \{d\}, F_2^{M_3} = F_3^{M_3} = \phi$ is a model for P_1. But $M_3 \le_1 M$ and $M_3 \neq M$, hence M is not minimal with respect to $\preceq_1$ and over all models of P_1.

In the following we consider an arbitrary monotonic operator η, instead of the operator ψ.

Definition 11 *Let η be a monotonic operator with respect to $\preceq_h$ and I an interpretation. We say that I is generated by η, if there is an interpretation I', such that $I = \eta^\omega(I')$.*

Let us denote by $\mathcal{M}^{P,\eta}$ the set of all models of P, which are generated by the operator η.

We extend the results of Theorem 2 on monotonic operators and consider the set of all models of P generated by these operators.

Theorem 4 *Let P be a positive program and η a monotonic operator with respect to $\preceq_h$. There exists the least fixed point of η, namely $\eta^\omega(\bot)$.*

If $\eta^\omega(\bot)$ is a model for P, then this model is the least model of P, with respect to $\preceq_h$ and on the set of all models of P generated by η.

The proof is straightforward.

The following example shows that, in general, the set of all models for P is different from the set of all models for P generated by the operator ψ.

Example 2 *Let P_2 be the following positive program ($n = 3$):*

$a \leftarrow b, c_{1/3}$
$a \leftarrow a, c_{2/3}$
$b \leftarrow a, c_{1/3}$
$c \leftarrow c_{2/3}$
$d \leftarrow c_{1/3}$

Let $I = (F_0^I, \ldots, F_3^I)$, where $F_0^I = F_3^I = \phi, F_1^I = \{d\}, F_2^I = \{c\}$.

I is a model for P_2, but it is not generated by the operator ψ as in Definition 5.

The following example points out that, in general, for a monotonic operator η, the interpretation $\eta^\omega(\bot)$ is not a model for P.

Example 3 *Let P_3 be the program as follows:*

$a \leftarrow b, c_{1/3}$
$a \leftarrow a, c_{2/3}$
$b \leftarrow a, c_{1/3}$
$c \leftarrow 1$
$d \leftarrow c_{1/3}$

Let $I = (F_0, \ldots, F_3)$ and η defined by: $\eta(I) = (G_0, \ldots, G_3)$, where $G_0 = \phi, G_1 = F_0, G_2 = F_1 \cup F_2, G_3 = F_3$. The operator η is monotonic with respect to $\preceq_1, \eta^\omega(\bot) = (\phi, \phi, H, \phi)$, but $\eta^\omega(\bot)$ is not a model for P_3.

Let $\Gamma^*(\eta, h, \rceil_h)$ be the operator defined for η, h and $\rceil_h$.

$$\Gamma^*(\eta, h, \rceil_h)(I) = M(P/I, \eta).$$

The difference between $\Gamma^*(\psi, h, \rceil_h)(I)$ and $\Gamma^*(\eta, h, \rceil_h)(I)$ consists of the fact that the first is a model for P, on the contrary the second is not, in general, a model for P.

Definition 12 *Every fixed point M of the operator $\Gamma^*(\eta, h, \rceil_h)$, in case it is a model for P/M, is called a multi-valued pseudo-stable model for P with respect to the operator η, the ordering $\preceq_h$ and the pseudo-negation $\rceil_h$.*

Theorem 5 *Let M be a multi-valued pseudo-stable model for P with respect to the operator η, the ordering $\preceq_h$ and the pseudo-negation $\rceil_h$.*

If M has the properties (P_1) and (P_2), then M is a minimal element of $\mathcal{M}_{P_1} \cap \mathcal{M}_{P_2} \cap \mathcal{M}^{P/M,\eta}$ with respect to the ordering $\rceil_h$.

The proof is similar with that of Theorem 3, consequently it is omitted.

6 Conclusion

In this paper we have proposed a pseudo-stable semantics for logic programs using pseudo-negations, partial orderings between interpretations and monotonic operators with respect to partial orderings. The class of pseudo-stable models for a program P contains strictly the class of 3-valued stable models for P. We have shown that if a multi-valued pseudo-stable model has certain properties, then it is a minimal model of P on the set of all models of P with those properties, with respect to the ordering $\preceq_h$.

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