Language, Liveness and Fairness Invariant in Decomposition of Petri Net Based on the Index of Place*

Qingtian Zeng

College of Information Science and Technology, Shandong University of Science and Technology, Qingdao 266510, P.R.China
{qtzeng@sdust.edu.cn}

Abstract It is not easy to analyze the physical system with Petri net if the system is large-scaled and complex, mainly because the structure of the Petri net model is also complex. In order to overcome this difficulty, a decomposition method of Petri net based on an index function of places is introduced, with which a set of well-formed and structure-simple sub net systems can be obtained. For all the sub net systems, the number of all the input or output places of each transition is less than or equal to one. The relationships about the reachable states, languages, liveness and fairness between the original Petri net and its sub net systems are analyzed with details. The sufficient and necessary conditions are presented to keep the reachable states and languages invariant during the decomposition process, which is useful for analysis of liveness and fairness of structure-complex Petri net systems.

Keywords Petri net, decomposition, index of places, Petri net language, reachable state, liveness, fairness

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1 Introduction

As models of modeling and analyzing physical systems, Petri net [1-3] has shown its abilities to deal with concurrency and conflict. In the last years, Petri net has been applied in many areas including computer network, software engineering, knowledge representation and verification, FMS (Flexible Manufacture System), DEDS (Discrete Event Dynamic System) and so on [4-7]. Petri net is used for the context of workflow management in which Petri net is an established tool for modeling and analyzing processes [4]. A high-level Petri net is introduced to model for modeling and verification of domain knowledge in current engineering, with which we can detect the abnormities, such as redundancy, conflict of the knowledge base effectually [5]. Petri nets also have been widely used to model, analyze and control the FMS [6, 7], since the elements of FMS can be represented by the places or transitions of Petri net model, and most of FMS concepts can be expressed by Petri net models.

When the physical system is complex, the Petri net model used for modeling the system is also structure-complex. Thus, it becomes difficult to analyze the complex physical system with Petri nets. In order to overcome this kind of difficulties, many solutions including decomposition, reduction, and composition for Petri net are introduced by the many researchers [8-13]. The synchronous composition of Petri nets is introduced and is used to analyze the state and behavior properties during the composition process, and the conditions to keep states and behaviors invariant are presented [8]. Recently, the concept of synchronous composition is extended from two Petri nets to more than three Petri nets, and an algorithm to obtain the language expression of structure-complex Petri net is presented [9]. Several generalized reduction methods for Petri net and a method for stepwise refinement and abstraction of Petri net are introduced respectively in [10] and [11]. A new concept of concurrent regular expression is introduced and a decomposition method is presented for obtaining the concurrent regular expressions of Petri Nets in [12]. The union decomposition of Petri net is introduced and the structure properties about the decomposition method are discussed [13]. It gives the conditions to keep the structure properties invariant in [13], but the dynamical properties don't be analyzed which is more important than structure properties analysis for the real physical systems.

We have introduced a new decomposition method for Petri net, with which a structure-complex Petri net is decomposed into some structure-simple nets such that $|{}^{\bullet}t|\le 1$ and $|t^{\bullet}|\le 1$ for all transitions [14]. With this

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method, the language characteristics of the decomposed nets can be analyzed and expressed easily [15]. In this paper, we have a further discussion on the decomposition method. The relationships about reachable states and firing transition sequences between the original Petri net and its sub net systems are analyzed with details, which is useful for analyzing the liveness and fairness of the original system. The sufficient and necessary conditions to keep the reachable states and languages invariant during the decomposition process are obtained, so the structure-simple nets obtained can be regarded as bridges to analyze and control the properties of the original structure-complex Petri net.

This paper is organized as follows. Section 2 presents the decomposition method for Petri net based on the index of the place. The language, liveness and fairness are analyzed during the decomposition process in section 3. Section 4 presents the sufficient and necessary conditions to keep the reachable states and language invariant. Section 5 concludes the whole paper.

2 Decomposition Method of Petri Net Based on the Index of Places

To save space, it is assumed that the readers are familiar with the basic concepts of Petri net [1, 2, and 3]. Some of the essential terminology and notations are defined as follows.

A Petri net is denoted as $\Sigma=(P,T;F,M_0)$, where $(P,T;F)$ is a net, and M_0 is the initial markings. If there is a $\sigma \in T^*$ such that $M_0[\sigma > M$, we say M is reachable from M_0. The set of all the reachable states from M_0 is denoted as $R(M_0)$. For any $t \in T$ and any $\sigma \in T^*$, $\#(t,\sigma)$ is the number of t appearing in σ. For any $t_1,t_2 \in T$, if there is an integer k satisfying that $\forall M \in R(M_0)$, $\forall \sigma \in T^*$: $M_0[\sigma >$ such that $\#(t_i,\sigma)=0 \rightarrow \#(t_j,\sigma) \le k$, $(i,j \in \{1,2\} \wedge i \ne j)$ we say that t_1 and t_2 satisfy fair relation in Σ. $\forall t_1,t_2 \in T$, if t_1 and t_2 hold fair relation in Σ, we say Σ is a fair Petri net.

In this section, we introduce a decomposition method, with which a structure-complex net can be decomposed into a set of structure-simple sub net systems such that $|{}^\bullet t| \le 1$ and $|t^\bullet| \le 1$ for all transitions.

Definition 1 Let $\Sigma=(P,T;F,M_0)$ be a Petri net. A function $f: P \rightarrow \{1,2,\ldots,k\}$ is named as an index function defined on the place set, for any $p_1,p_2 \in P$, $(p_1^\bullet \cap p_2^\bullet \ne \phi) \vee ({}^\bullet p_1 \cap {}^\bullet p_2 \ne \phi) \rightarrow f(p_1) \ne f(p_2)$, and $f(p)$ is named as index of place p.

Definition 2 Let $\Sigma=(P,T;F,M_0)$ be a Petri net. $\forall M \in R(M_0)$, and $P_1 \subseteq P$, the projection of M on P_1, denoted as $\Gamma_{P\rightarrow P_1}(M)$, is defined as $\forall p \in P_1$, $\Gamma_{P\rightarrow P_1}(M)(p)=M(p)$.

Definition 3[14] Let $\Sigma=(P,T;F,M_0)$ be a Petri net, and $f: P \rightarrow \{1,2,\ldots,k\}$ be the index function on the place set of Σ. Petri net $\Sigma_i=(P_i,T_i;F_i,M_{0i})$ ($i \in \{1,2,\ldots,k\}$) is said to be the decomposition net systems of Σ based on f if the following conditions are satisfied.

(1) $P_i=\{p \in P \mid f(p)=i\}$;
(2) $T_i=\{t \in T \mid \exists p \in P_i, t \in {}^\bullet p \cup p^\bullet\}$;
(3) $F_i=\{(P_i \times T_i) \cup (T_i \times P_i)\} \cap F$;
(4) $M_{0i}=\Gamma_{P\rightarrow P_i} M_0$;

Simply, we say Σ_i is the index decomposition net of Σ.

Example 1 A Petri net $\Sigma=(P,T;F,M_0)$ is shown in Fig 1. The index function $f: P \rightarrow \{1,2,\ldots,k\}$ can be defined as $f(p_1)=f(p_4)=1$, $f(p_3)=f(p_6)=2$, and $f(p_2)=f(p_5)=3$. It is obvious that f satisfies the conditions in Definition 1. Based on f, Σ is decomposed into three structure-simple nets Σ_1, Σ_2 and Σ_3 shown in Fig. 2.

With Definition 3, the results of the decomposition method are not unique. If we choose a different index function f satisfying the conditions in Definition 1, the indexes of the places are different, so the decomposition result maybe is different. In all cases, two or more input (or output) places of the same transition can not have the same index value, thus the structure-complex Petri net can be decomposed into a set of structure-simple Petri nets. In order to analyze the properties expediently, the minimal value of all k

satisfying the conditions in Definition 1 is regarded the best condition to define the index function f. More discussions about the decomposition method can be seen in [14].

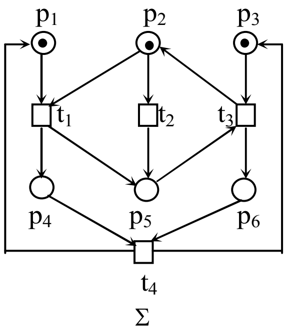

Fig.1. Petri Net Σ

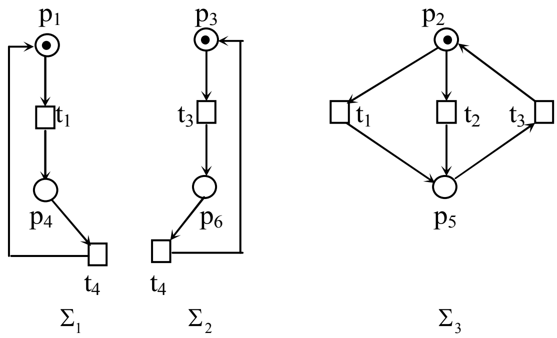

Fig.2. Three Sub-net Systems of Σ

Theorem 1 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i})$ ($i \in \{1, 2, \ldots, k\}$) be the index decomposition nets of $\Sigma = (P, T; F, M_0)$. $\forall t \in T_i$, $|{}^{\bullet}t| \le 1$ and $|t^{\bullet}| \le 1$.

Proof. It can be easily to prove with Definition 3.

Theorem 1 indicates that all index decomposition sub net systems are structure-simple. A Petri net $\Sigma = (P, T; F, M_0)$ is an S-Net if $\forall t \in T_i$ such that $|{}^{\bullet}t| \le 1$ and $|t^{\bullet}| \le 1$ [16]. So, each of the index decomposition nets is an S-Net. Since the Structure of S-Net is simple, it is easy to express its language and analyze its liveness [16, 17].

3 System Properties Analysis with the Decomposition

In this section, we discuss the relationships about languages (transition firing sequence), liveness and fairness between the sub nets systems and the original system during the decomposition process so as to show it is useful for property analysis of structure-complex system.

3.1 Languages

In the theory of Petri net languages, four different types named L-type, G-type, T-type and P-type are defined based on the difference of the set of the reachable states[1]. Depending on the choice of transition labeling function (free, λ-free, arbitrary), each type is divided into three kinds.

In this paper, we use free L-type language as example to discuss. The reachable state set of a Petri net $\Sigma = (P, T; F, M_0)$ is defined as $Q_T \subseteq R(M_0) \wedge (\forall M_e \in Q_T, \forall p \in P - P_f : M_e(p) = 0)$, where $P_f \subseteq P$ is a given place set, and named as the end place set of Σ..

Definition 4 Let $\Sigma = (P, T; F, M_0)$ be a Petri net and $P_f \subseteq P$. $L(\Sigma)$ is the language of Σ iff $L(\Sigma) = \{\sigma \mid \sigma \in T^* \wedge M_0[\sigma > M_e \wedge (\forall p \in P - P_f, M_e(p) = 0)\}$.

In the following discussions, a Petri net is denoted as $\Sigma = (P, T; F, M_0, P_f)$ where P_f is the end place set of Σ. If $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1, 2, \cdots, k\}$) is the index decomposition net systems of $\Sigma = (P, T; F, M_0, P_f)$, Σ_i holds the conditions of Definition 3 and $P_{fi} = P_f \cap P_i$.

Definition 5 Let X be a finite alphabet and $Y \subseteq X$. The projection from X to Y, $\Pi_{X \to Y}$ is defined as for all $\sigma \in X^*$, $\Pi_{X \to Y}\sigma$ is the sub-string of σ with deleting all $a \in (X - Y)$ from σ.

Definition 6 Let $\Sigma = (P, T; F, M_0, P_f)$ be a Petri net. $\forall M \in R(M_0)$, for $P_1 \subseteq P$, the projection of M over P_1, denoted by $\Gamma_{P \to P_1}(M)$, is defined as $\forall p \in P_1$ such that $\Gamma_{P \to P_1}(M)(p) = M(p)$.

Theorem 2[15] If $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1, 2, \ldots, k\}$) is the index decomposition net systems of Petri net $\Sigma = (P, T; F, M_0, P_f)$, then $\forall M \in R(M_0)$, $\Gamma_{P \to P_i}(M) \in R(M_{0i})$ ($i \in \{1, 2, \ldots, k\}$).

Proof. See [15].

Theorem 3 If Petri net $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$

($i \in \{1,2,\ldots,k\}$) is the index decomposition net systems of Petri net $\Sigma = (P,T;F,M_0,P_f)$, for any $\sigma \in L(\Sigma)$, $\Pi_{T\to T_i}(\sigma) \in L(\Sigma_i)$. Tthat is $\Pi_{T\to T_i}(L(\Sigma)) \subseteq L(\Sigma_i)$.

Proof. With Theorem 2, it can follow easily.

Definition 7 Let $L(\Sigma_i)$ be the language of Petri net $\Sigma_i = (P_i,T_i;F_i,M_{0i},P_{fi})$ ($i \in \{1,2\}$), $L = L(\Sigma_1)[]L(\Sigma_2)$ is the synchronous intersection language of $L(\Sigma_1)$ and $L(\Sigma_2)$, if $\forall \sigma \in T^* \wedge \sigma \in L$ where $T = T_1 \cup T_2$, $\sigma_i = (\Pi_{T\to T_i}\sigma) \wedge \sigma_i \in L(\Sigma_i)$.

$L = L(\Sigma_1)[]L(\Sigma_2)$ can also be denoted as $L = \overset{2}{\underset{i=1}{[]}} L(\Sigma_i)$. Similarly, the synchronous intersection language of $L(\Sigma_i)(k \geq 3)$ and $i \in \{1,2,\ldots,k\}$ can be defined as $L(\Sigma) = (\overset{k-1}{\underset{i=1}{[]}} L(\Sigma_i))[]L(\Sigma_k) = \overset{k}{\underset{i=1}{[]}} L(\Sigma_i)$.

With Definition 7, it can be proved that $L(\Sigma_1)[]L(\Sigma_2) = L(\Sigma_1) \cap L(\Sigma_2)$ iff $T_1 = T_2$. So, the synchronous intersection operation is different from the interaction operation of languages. The interaction operation is only a special case of the synchronous intersection operation while $T_1 = T_2$.

Theorem 4 If Petri net $\Sigma_i = (P_i,T_i;F_i,M_{0i},P_{fi})$ ($i \in \{1,2,\ldots,k\}$) is the index decomposition net systems of $\Sigma = (P,T;F,M_0,P_f)$, then $L(\Sigma) = \overset{k}{\underset{i=1}{[]}} L(\Sigma_i)$.

Proof. Let $M_f \subseteq R(M_0)$, and $\forall M \in M_f$ such that $\forall p \in P - P_f, M(p) = 0$. Then $M_{fi} = \{\Gamma_{P\to P_i} M \mid M \in M_f\}$, and $L(\Sigma) = \{\sigma \mid (\sigma \in T^*) \wedge (M_0[\sigma > M) \wedge (M \in M_f)\}$.

Now we use induction on $|\sigma|$ to prove the conclusion.

(1)If $|\sigma| = 1$, then $\sigma_i = \Pi_{T\to T_i}(\sigma) = \begin{cases} \sigma & \sigma \in T_i \\ \varepsilon & \sigma \notin T_i \end{cases}$ ($i \in \{1,2,\ldots,k\}$). With $\sigma \in L(\Sigma)$ iff $M_0[\sigma > M_1 \wedge M_1 \in M_f$, and iff $\Gamma_{P\to P_i}(M_0)[\sigma_i > \Gamma_{P\to P_i}(M_1) \wedge \Gamma_{P\to P_i}(M_1) \in M_{fi}$, so $\sigma_i \in L(\Sigma_i)$ ($i \in \{1,2,\ldots,k\}$).

(2)Suppose that the conclusion is correct while $|\sigma| = n$. We prove that it is also correct while $|\sigma| = n+1$.

Let $\sigma = \sigma' \bullet t'$, where $|\sigma'| = n$, and t' is the $(n+1)th$ element of σ.

$\sigma \in L(\Sigma)$ iff $M_0[\sigma' \bullet t' > M_{n+1} \wedge M_{n+1} \in M_f$. Let $M_0[\sigma' > M_n[t' > M_{n+1}$, then $M_n \in R(M_0)$, and $M_f^{'} = \{M_n \mid M_0[\sigma' > M_n[t' > M_{n+1} \wedge M_{n+1} \in M_f\}$, and $M_{fi}^{'} = \{\Gamma_{P\to P_i} M_n \mid M_n \in M_f^{'}\}$.

With the supposition, $\exists \sigma_i^{'} = \Pi_{T\to T_i}(\sigma')$ and $\sigma_i^{'} \in L(\Sigma_i)$ such that $\sigma_i = \Pi_{T\to T_i}(\sigma) = \begin{cases} \sigma_i^{'} \bullet t' & t' \in T_i \\ \sigma_i^{'} & t' \notin T_i \end{cases}$ and $\sigma_i \in L(\Sigma_i)$ iff $M_{0i}[\sigma_i^{'} > M_i^{'} \wedge M_i^{'} \in M_{fi}^{'}$, and iff $M_{0i}[\sigma_i > M_{(n+1)i} \wedge M_{(n+1)i} \in M_{fi}$. So, $\sigma_i \in L(\Sigma_i)$ and $\sigma_i = \Pi_{T\to T_i}(\sigma)$.

Thus, the theorem has been proved.

Theorem 4 shows that $\overset{k}{\underset{i=1}{[]}} L(\Sigma_i)$ is the language expression of Σ, if the language expression $L(\Sigma_i)$ ($i \in \{1,2,\ldots,k\}$) can be obtained. Theorem 1 shows that each Σ_i is an S-net. Since the structure of the S-Net is simple, it is easy to present the language expression for each S-Net. The methods to obtain the language expressions of all kinds of S-Nets have been presented in [16]. With the above analysis, a method to obtain the language expression for a Petri net especially a structure-complex Petri net is obtained.

Take the Petri net Σ shown in Fig.1 as an example to obtain its language expression. Suppose that $P_f = \{p_1, p_2, p_3\}$, which can be decided by users according to the states of the physical system. With the methods in [16], the language expressions of Σ_1, Σ_2 and Σ_3 can be expressed as $L(\Sigma_1) = (t_1 t_4)^*$, $L(\Sigma_2) = (t_3 t_4)^*$, and $L(\Sigma_3) = ((t_1 + t_2)t_3)^*$. With Theorem 4, the language expression of Σ is $L(\Sigma) = L(\Sigma_1)[]L(\Sigma_2)[]L(\Sigma_3) = (t_1 t_4)^*[](t_3 t_4)^*[]((t_1 + t_2)t_3)^*$.

3.2 Liveness

Definition 8[18] Petri net $\Sigma = (P,T;F,M_0,P_f)$ is live iff $\forall t \in T$, $\forall M \in R(M_0)$, $\exists M^{'} \in R(M)$ such that $M^{'}[t >$.

Definition 9[18] Let L be the language of alphabet T. $\forall \sigma \in L$, the set $\&(\sigma) = \{t \mid t \in T \wedge \#(t,\sigma) \geq 1\}$ is the set of characters appearing in σ. The strong closed language of L is defined as $\vec{L} = \{\sigma \in T^* \mid \exists \sigma' \in T^* : \sigma \bullet \sigma' \in L \wedge \&(\sigma') = T\}$.

Lemma 1 Petri net $\Sigma = (P, T; F, M_0, P_f)$ is live iff $\vec{L}(\Sigma) = L(\Sigma)$.

Proof. See the Theorem 2 in [18].

Theorem 5 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1, 2, \ldots, k\}$) be the index decomposition net systems of $\Sigma = (P, T; F, M_0, P_f)$. If $\Pi_{T \to T_i}(L(\Sigma)) = L(\Sigma_i)$, then Σ is live iff Σ_i is live.

Proof.

(1) $\forall \sigma_i \in L(\Sigma_i)$, since $\Pi_{T \to T_i}(L(\Sigma)) = L(\Sigma_i)$ ($i \in \{1, 2, \ldots, k\}$), there must $\exists \sigma \in L(\Sigma)$ such that $\Pi_{T \to T_i} \sigma = \sigma_i$. With Lemma 1, for $\forall \sigma \in L(\Sigma)$, there $\exists \sigma' \in T^*$ such that $\sigma \bullet \sigma' \in L(\Sigma) \wedge \&(\sigma') = T$ since Σ is live,. So, $\Pi_{T \to T_i}(\sigma \bullet \sigma') = (\Pi_{T \to T_i} \sigma) \bullet (\Pi_{T \to T_i} \bullet \sigma') = \sigma_i \bullet \sigma_i'$ and $\sigma_i \bullet \sigma_i' \in L(\Sigma_i)$.

$\&(\sigma_i') = \&(\Pi_{T \to T_i}(\sigma')) = \Pi_{T \to T_i} \& (\sigma') = \Pi_{T \to T_i} T = T_i$ ($i \in \{1, 2, \ldots, k\}$), so Σ_i ($i \in \{1, 2, \ldots, k\}$) is live with Lemma 1.

(2) $\forall \sigma \in L(\Sigma)$, and let $\sigma_i = \Pi_{T \to T_i}(\sigma)$ ($i \in \{1, 2, \ldots, k\}$). With Theorem 3, $\sigma_i \in L(\Sigma_i)$. With Lemma 1, there $\exists \sigma_i' \in T_i^*$ such that $\sigma_i \bullet \sigma_i' \in L(\Sigma_i)$ since Σ_i is live,. Because $\Pi_{T \to T_i}(L(\Sigma)) = L(\Sigma_i)$ ($i \in \{1, 2, \ldots, k\}$), $\sigma_i \bullet \sigma_i' \in L(\Sigma_i)$ and $\exists \sigma' \in L(\Sigma)$ such that $\Pi_{T \to T_i}(\sigma') = \sigma_i \bullet \sigma_i'$. Since $\sigma_i = \Pi_{T \to T_i}(\sigma)$ and $\sigma \in L(\Sigma)$, $\exists \sigma' \in T^*$ such that $\sigma \bullet \sigma' \in L(\Sigma)$, and since $\&(\sigma_i') = T_i$ and $\&(\Pi_{T \to T_i}(\sigma')) = \&(\sigma_i)$, $\&(\sigma') = T$. With Lemma 1, Σ is live.

3.3 Fairness

Definition 10[19] Petri net $\Sigma = (P, T; F, M_0)$ is fair iff $\forall t_1, t_2 \in T$, $\exists k > 0$, $\forall M \in R(M_0)$, $\forall \sigma \in T^*$ such that $M[\sigma > \wedge (\#(t_i / \sigma) = 0 \to \#(t_j / \sigma) < k)$, ($i, j \in \{1, 2\}, i \neq j$).

Theorem 6 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1, 2, \ldots, k\}$) be the index decomposition net systems of $\Sigma = (P, T; F, M_0, P_f)$. If Σ_i is fair, then Σ is fair.

Proof. Let $RMG(\Sigma)$ be the reachability graph of Σ, and $RMG(\Sigma_i)$ be the reachability graph of Σ_i ($i \in \{1, 2, \ldots, k\}$). With Theorem 2, $\Pi_{T \to T_i}(L(\Sigma)) \subseteq L(\Sigma_i)$. $\forall t \in T$, if $\exists M \in R(M_0)$ such that $M[t >$ and $t \in T_i$, then $\Gamma_{P \to P_i}(M)[t >$. Since Σ_i is fair, with the Theorem 5 in [19], for any directed loop C_i in $RMG(\Sigma_i)$, $\forall t \in T_i$, there must be an edge in C_i marked with t. So, for any directed loop C in $RMG(\Sigma)$, $\forall t \in T$, there must be an edge in C marked with t, Σ is proved to be fair.

Theorem 7 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1, 2, \ldots, k\}$) be the index decomposition net systems of $\Sigma = (P, T; F, M_0, P_f)$. If Σ is live and fair and $\Pi_{T \to T_i}(L(\Sigma)) = L(\Sigma_i)$, then Σ_i is live and fair.

Proof. Since $\Pi_{T \to T_i}(L(\Sigma)) = L(\Sigma_i)$ ($i \in \{1, 2, \ldots, k\}$) and Σ is live, with Theorem 5, Σ_i ($i \in \{1, 2, \ldots, k\}$) is also live. So there are no output branches in $RMG(\Sigma)$ and $RMG(\Sigma_i)$, and for any loop in $RMG(\Sigma_i)$, there must be a responding loop in $RMG(\Sigma)$. Since Σ is fair, with the Theorem 5 in [19], Σ_i ($i \in \{1, 2, \ldots, k\}$) is fair.

4 Conditions for Properties Invariant of the Decomposition

With the liveness and fairness analysis during the decomposition process, the condition $\Pi_{T \to T_i}(L(\Sigma)) = L(\Sigma_i)$ is only necessary to keep the properties of the original system invariant. However, With Theorem 2 and Theorem 3, we know that $\Gamma_{P \to P_i}(M) \in R(M_{0i})$ and $\Pi_{T \to T_i}(L(\Sigma)) \subseteq L(\Sigma_i)$.

In this section, we show how to determine whether the conditions are satisfied during the decomposition, and present the sufficient and necessary conditions for $\Pi_{T \to T_i}(L(\Sigma)) = L(\Sigma_i)$ (and $\Gamma_{P \to P_i}(R(M_0)) = R(M_{0i})$). Based on the reachability graph, the method is presented to decide whether $\Pi_{T \to T_i}(L(\Sigma)) = L(\Sigma_i)$ (and $\Gamma_{P \to P_i}(R(M_0)) = R(M_{0i})$).

Theorem 8 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1, 2, \ldots, k\}$) be the index decomposition net systems of Petri net $\Sigma = (P, T; F, M_0, P_f)$. $\Pi_{T\to T_i}(L(\Sigma)) = L(\Sigma_i)$ iff

$\Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i)) = \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$ for all $i, j \in \{1, 2, ..., k\}$, $i \neq j$ and $T_i \cap T_j \neq \phi$.

Proof. (1) First we prove $\Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i)) = \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$ if $\Pi_{T\to T_i}(L(\Sigma)) = L(\Sigma_i)$ for all $i, j \in \{1, 2, ..., k\}$, $i \neq j$ and $T_i \cap T_j \neq \phi$. And, it is first to prove $\Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i)) \subseteq \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$.

Let $T_\Delta = T_i \cap T_j \neq \phi$ and $\forall \sigma_{\Delta i} \in \Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i))$, then $\exists \sigma_i \in L(\Sigma_i)$ such that $\Pi_{T_i \to T_i \cap T_j}(\sigma_i) = \sigma_{\Delta i}$.

Since $\Pi_{T\to T_i}(L(\Sigma)) = L(\Sigma_i)$ ($i \in \{1, 2, \ldots, k\}$), for any $\sigma_i \in L(\Sigma_i)$, $\exists \sigma \in L(\Sigma)$ such that $\Pi_{T\to T_i}(\sigma) = \sigma_i$.

Thus, $\sigma_{\Delta i} = \Pi_{T_i \to T_i \cap T_j}(\sigma_i) = \Pi_{T_i \to T_i \cap T_j}(\Pi_{T\to T_i}(\sigma)) = \Pi_{T \to T_i \cap T_j}(\sigma) \in \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$.

So, $\Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i)) \subseteq \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$.

Similarly, $\Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j)) \subseteq \Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i))$ can be proved.

Therefore $\Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i)) = \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$.

(2)Then, we prove $\Pi_{T\to T_i}(L(\Sigma)) = L(\Sigma_i)$ if $\Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i)) = \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$ for all $i, j \in \{1, 2, ..., k\}$, $i \neq j$ and $T_i \cap T_j \neq \phi$.

$\forall \sigma_i \in L(\Sigma_i)$, let $\sigma_{\Delta i} = \Pi_{T_i \to T_i \cap T_j}(\sigma_i)$. Since $\Pi_{T_i \to T_i \cap T_j}(L(\Sigma_i)) = \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$, $\sigma_{\Delta i} \in \Pi_{T_j \to T_i \cap T_j}(L(\Sigma_j))$. That is $\exists \sigma_j \in L(\Sigma_j)$ such that $\Pi_{T_j \to T_i \cap T_j}(\sigma_j) = \sigma_{\Delta i}$.

Let $\sigma = \{\omega \mid \omega \in T^* \wedge (\Pi_{T\to T_i}(\omega) = \sigma_i) \wedge (\Pi_{T\to T_j}(\omega) = \sigma_j)\}$ it is easy to prove

$\sigma \in \Pi^{-1}_{T_i \to T}(L(\Sigma_i)) \cap \Pi^{-1}_{T_j \to T}(L(\Sigma_j)) = L(\Sigma)$.

It is obvious that $\Pi_{T\to T_i}(\sigma) = \sigma_i \in L(\Sigma_i)$ and $\Pi_{T\to T_j}(\sigma) = \sigma_j \in L(\Sigma_j)$, so $L(\Sigma_i) \subseteq \Pi_{T\to T_i}(L(\Sigma))$.

Similarly, $L(\Sigma_j) \subseteq \Pi_{T\to T_j}(L(\Sigma))$, $L(\Sigma_l) \subseteq \Pi_{T\to T_l}(L(\Sigma))$ ($l \in \{1, 2, \ldots, k\}$) can be proved.

Based on Theorem 3, $\Pi_{T\to T_i}(L(\Sigma)) \subseteq L(\Sigma_i)$ ($i \in \{1, 2, \ldots, k\}$).

Thus, $\Pi_{T\to T_i}(L(\Sigma)) = L(\Sigma_i)$ ($i \in \{1, 2, \ldots, k\}$).

With the conditions to keep the projection of the language invariant on the subsets, we can obtain the conditions to keep the projection of the reachable states invariant on the subsets presented in Theorem 9.

Theorem 9 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1, 2, \ldots, k\}$) be the index decomposition net systems of Petri net $\Sigma = (P, T; F, M_0, P_f)$. $\Gamma_{P\to P_i}(R(M_0)) = R(M_{0i})$ iff $\Pi_{T\to T_i}(L(\Sigma)) = L(\Sigma_i)$.

Proof. With Theorem 8, it is easy to prove.

Next, we present the methods to decide whether $\Pi_{T\to T_i}(L(\Sigma)) = L(\Sigma_i)$ (and $\Gamma_{P\to P_i}(R(M_0)) = R(M_{0i})$) based on the reachability graph.

Definition 11 Let $G_i = < V_i, E_i; f_i >$ ($i = 1, 2$) be two marked directed graphs, where V_i, E_i are the vertex set and edge set, and $f_i : E_i \to \Omega_i$ is a one-to-one mapping from E_i to an alphabet Ω_i. G_1 and G_2 are isomorphic about non-empty set Ω_Δ, denoted as $\Phi_{\Omega_\Delta}(G_1) \cong \Phi_{\Omega_\Delta}(G_2)$, iff

(1) $f_1(E_1) \cap f_2(E_2) = \Omega_\Delta$;

(2)there is $V_{\Delta i} \subseteq V_i (i = 1, 2)$ such that $|V_{\Delta 1}| = |V_{\Delta 2}|$ and there is a one-to-one mapping from $V_{\Delta 1}$ to $V_{\Delta 2}$;

(3)if $v_{xi}, v_{yi} \in V_{\Delta i}$, $(v_{xi}, v_{yi}) \in E_i (i = 1, 2)$, $v_{x1} \leftrightarrow v_{x2}$, and $v_{y1} \leftrightarrow v_{y2}$, then $f(v_{x1}, v_{y1}) = f(v_{x2}, v_{y2})$.

The reachability graph $RMG(\Sigma)$ of a Petri net $\Sigma = (P, T; F, M_0, P_f)$ is a marked directed graph [1], where $V = R(M_0)$, $\Omega = T$ and $(M_1, M_2) \in E$ iff $\exists t \in T$ such that $M_1[t > M_2$, and we define $f((M_1, M_2)) = t$.

Theorem 10 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i = 1, 2$) be two Petri nets such that $T_1 \cap T_2 = T_\Delta \neq \phi$, and $RMG(\Sigma_i)$ be the reachability graph of Σ_i. $\Pi_{T_1 \to T_\Delta}(L(\Sigma_1)) = \Pi_{T_2 \to T_\Delta}(L(\Sigma_2))$ iff $\Phi_{T_\Delta}(RMG(\Sigma_1)) \cong \Phi_{T_\Delta}(RMG(\Sigma_2))$.

Proof. Since $L(RMG(\Sigma_i)) = L(\Sigma_i)$ ($i = 1, 2$), and the action simplifying $RMG(\Sigma_i)$ into $\Phi_{T_\Delta}(RMG(\Sigma_i))$ is equal to $\Pi_{T_i \to T_\Delta}(L(\Sigma_i))$ acting on the language, we can obtain $L(\Phi_{T_\Delta}(RMG(\Sigma_1))) = \Pi_{T_i \to T_\Delta}(L(\Sigma_i))$. When simplifying $RMG(\Sigma_i)$ into $\Phi_{T_\Delta}(RMG(\Sigma_i))$, if edges in $RMG(\Sigma_i)$ are not in $\Phi_{T_\Delta}(RMG(\Sigma_i))$, the marked denotations are regarded as empty letters. So, $\Pi_{T_1 \to T_\Delta}(L(\Sigma_1)) = \Pi_{T_2 \to T_\Delta}(L(\Sigma_2))$ iff

$\Phi_{T_\Delta}(RMG(\Sigma_1)) \cong \Phi_{T_\Delta}(RMG(\Sigma_2))$.

Following the related conclusions obtained in the former sections, four corollaries can be obtained.

Corollary 1 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1,2,\ldots,k\}$) be the index decomposition net systems of $\Sigma = (P,T;F,M_0,P_f)$, and $RMG(\Sigma_i)$ be the reachability graph of Σ_i. $\Pi_{T\to T_i}(L(\Sigma)) = L(\Sigma_i)$ iff for $\forall i,j \in \{1,2,...,k\}$, if $T_i \cap T_j = T_\Delta \neq \phi$, then $\Phi_{T_\Delta}(RMG(\Sigma_i)) \cong \Phi_{T_\Delta}(RMG(\Sigma_j))$.

Corollary 2 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1,2,\ldots,k\}$) be the index decomposition net systems of $\Sigma = (P,T;F,M_0,P_f)$, and $RMG(\Sigma_i)$ be the reachability graph of Σ_i. $\Gamma_{P\to P_i}(R(M_0)) = R(M_{0i})$ iff for $\forall i,j \in \{1,2,...,k\}$, if $T_i \cap T_j = T_\Delta \neq \phi$, then $\Phi_{T_\Delta}(RMG(\Sigma_i)) \cong \Phi_{T_\Delta}(RMG(\Sigma_j))$.

Corollary 3 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1,2,\ldots,k\}$) be the index decomposition net systems of $\Sigma = (P,T;F,M_0,P_f)$, and $RMG(\Sigma_i)$ be the reachability graph of Σ_i. If for $\forall i,j \in \{1,2,...,k\}$, $T_i \cap T_j = T_\Delta \neq \phi$ and $\Phi_{T_\Delta}(RMG(\Sigma_i)) \cong \Phi_{T_\Delta}(RMG(\Sigma_j))$, then Σ is live iff Σ_i is live.

Corollary 4 Let $\Sigma_i = (P_i, T_i; F_i, M_{0i}, P_{fi})$ ($i \in \{1,2,\ldots,k\}$) be the index decomposition net systems of $\Sigma = (P,T;F,M_0,P_f)$, and $RMG(\Sigma_i)$ be the reachability graph of Σ_i. If for $\forall i,j \in \{1,2,...,k\}$, $T_i \cap T_j = T_\Delta \neq \phi$ and $\Phi_{T_\Delta}(RMG(\Sigma_i)) \cong \Phi_{T_\Delta}(RMG(\Sigma_j))$, then Σ is live and fair iff Σ_i is live and fair.

We take the Petri net in Fig 1 as an example to show the analysis above. With the method to construct the rachability graph of Petri net [1], the reachability graphs of the three subnet systems are shown in Fig 3.

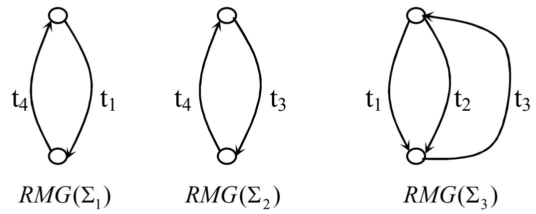

Fig 3 The Reachability Graphs of Three Sub-net Systems

In Fig. 3, since $T_1 \cap T_2 = \{t_4\}$, and $RMG(\Sigma_1)$ and $RMG(\Sigma_2)$ are isomorphic about $\{t_4\}$. Since $T_1 \cap T_3 = \{t_1\}$, $RMG(\Sigma_1)$ and $RMG(\Sigma_2)$ are isomorphic about $\{t_1\}$. Similarly $T_2 \cap T_3 = \{t_3\}$, $RMG(\Sigma_2)$ and $RMG(\Sigma_3)$ are isomorphic about $\{t_3\}$. We can easily show that Σ is fair and live, so Σ_1, Σ_2, and Σ_3 are also fair and live.

5 Conclusions

There are three main contributions of this paper to the Petri net theory and its applications. (1) It introduces a decomposition method for structure-complex Petri net based on the index function on the place set, with which the structure-complex is decomposed into a set of structure-simple net systems such that $|{}^\bullet t| \le 1$ and $|t^\bullet| \le 1$ for all transitions. Since the structure is simple, the language of each decomposition net system can be analyzed and expressed easily. (2) The relationships about reachable states, languages, fairness and liveness between the original system and its decomposition net systems are analyzed with details. (3) The necessary and sufficient conditions to keep the reachable states and firing sequences invariant are presented.

In the future, we will continue to analyze the language relationships between the language expression containing synchronous intersection and the traditional language expression [20]. And, we will have researches on the analysis of the real physical system based on the language expression obtained with the method of this paper.

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